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Gabor frames from contact geometry in models of the primary visual cortex.

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Abstract

We introduce a model of the primary visual cortex (V_1), which allows the compression and decomposition of a signal by a discrete family of orientation and position dependent receptive profiles. We show in particular that a specific framed sampling set and an associated Gabor system is determined by the Legendrian circle bundle structure of the 3-manifold of contact elements on a surface (which models the V_1-cortex), together with the presence of an almost complex structure on the tangent bundle of the surface (which models the retinal surface). We identify a maximal area of the signal planes, determined by the retinal surface, that provides a finite number of receptive profiles, sufficient for good encoding and decoding. We then consider a 5-dimensional model where receptive profiles also involve a dependence on frequency and scale variables, in addition to the dependence of position and orientation. In this case we show that the proposed window function does not give rise to frames (even in a distributional sense), while a natural modification of the same window generates Gabor frames with respect to the appropriate lattice determined by the contact geometry.

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1 Introduction

Two interesting mathematical models account for the functional architecture of the V_1 visual cortex, both of them originally developed in the ’80s and considerably expanded and refined in recent years: a model of receptive profiles in terms of Gabor functions, [13, 14, 30], and a model of the connectivity and the hypercolumn structure of the V_1 cortex in terms of contact geometry and contact bundles [25]. These two aspects of the mathematical modeling of the visual cortex may appear at first unrelated, the first capturing functional analytic aspects of signal encoding in terms of the neurons receptive profiles, the latter describing the geometric structure of the visual cortex that captures the sensitivity to orientation of the simple cells in the hypercolumns. The fiber

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bundle contact geometry also provides a good geometric description of the connections between simple cells in different hypercolumns. These two mathematical models are in fact closely entangled, as the more recent works of Petitot and Tondut [34], Citti and Sarti [11], and Sarti et al. [37] have clearly shown. The simple cells profile shapes and their geometric arrangement in the hypercolumn structure are simultaneously governed by the same action of the rototranslation group, combined with a principle of selectivity of maximal response (see [11]). Thus, it appears that the contact geometry of the V_1 cortex also determines its signal analysis properties. This is an interesting mathematical observation in itself, that certain classes of contact manifolds carry an associated signal analysis framework entirely determined by the geometry. Part of the purpose of the present paper is to clarify what this means in the specific case of contact 3-manifolds that are Legendrian circle bundles over a surface, and contact 5-manifolds obtained from them by symplectization and contactization, which are the two cases of direct relevance to the neuroscience modeling. Our main focus here is on identifying additional aspects of the contact geometry that have a direct influence on the signal analysis properties, beyond the relations already identified in previous work such as [37]. In particular, while previous work focused on continuous representations of signals through short time Fourier transform, we argue that a more refined model should incorporate the discrete nature of the neuron population involved, and identify a mechanism that ensures a good signal encoding and decoding in terms of a selection of a discrete system of filters. We argue that this selection of a discrete Gabor system with adequate signal analysis properties can also be seen as directly encoded in the geometric model of the V_1-cortex. Our key observation to this purpose is the fact that the combined presence of the contact structure on the Legendrian circle bundle and a complex structure on the base surface determines an associated bundle of framed lattices, which in turn provide the required discrete sampling set for the Gabor frames.

Gabor filters play an essential role to both neural modeling and signal processing. In the works of Daugman [13, 14] and Marcelja [30], it is argued why Gabor filters are the right choice for the modeling of receptive profiles of visual neurons in V_1. In particular, simple cells of the primary visual cortex try to localize at the same time the position (x, y) and the frequency w of a signal detected in the retina. However, the uncertainty principle in signal analysis indicates that it is impossible to detect both position and frequency with arbitrary precision. Gabor filters minimize the uncertainty and therefore they process spatiotemporal information optimally. Thus, a receptive profile, centered at (x_0, y_0), with preferred spatial frequency $w = \sqrt{u_0^2 + v_0^2}$ and preferred orientation $\theta = \arctan(\frac{v_0}{u_0})$ is efficiently modelled by a bivariate, real-valued Gabor function $f(x, y)$ of the form $\exp(-\pi((x - x_0)^2 + (y - y_0)^2)) \exp(-2\pi i(u_0(x - x_0) + u_0(y - y_0)))$. Given a distribution $I(x, y)$, specifying the distribution of light intensity of a visual stimulus, the receptive profile generates the response to that distributed stimulus via integration

$$\text{response} = \int\int_{-\infty}^{+\infty} I(x, y) f(x, y) dx dy.$$

The integral representing the response of a receptive field is commonly used in time-frequency analysis, as short time Fourier transform. In a Euclidean space $\mathbb{R}^d$ of arbitrary dimension, the short time Fourier transform of a signal I with respect to a window function g is a linear and continuous, joint time-frequency representation defined as

$$V_g I(x, w) = \int_{\mathbb{R}^d} I(t)\overline{g(t - x)} e^{-2\pi i t \cdot w} dt, \quad \text{for} \quad x, w \in \mathbb{R}^d.$$

More specifically, in the plane $\mathbb{R}^2$, the response of a receptive profile to a visual signal I is equal to the short time Fourier transform of the signal I with respect to the Gaussian

$g(x,y) = \exp(-\pi(x^2+y^2))$, multiplied with a complex exponential

$$\text{response} = e^{2\pi i (x_0,y_0)\cdot(u_0,v_0)} V_g I(x_0,y_0,u_0,v_0).$$

The short time Fourier transform is suitable for most theoretical approaches of space-frequency and time-frequency analysis. However, it is not practical to use continuous representations for experimental purposes, when dealing with a finite (albeit large) population of neurons. Continuous representations of signals, like the short time Fourier transform, allow good encoding and decoding of the signal by using an uncountable system of receptive profiles. Methods from discrete time-frequency analysis come to solve this problem. In discrete methods, a discrete system of Gabor elementary functions is enough to reconstruct and deconstruct the signal. If the window function is supported on a subset of the ambient Euclidean space centered in (x,y), the STFT $V_g I(x_0,y_0,u_0,v_0)$ carries the same information for neighbouring points in the support of g and therefore it is possible to reduce the sampling set without compromising the quality of encoding and decoding of the signal. While there is rich bibliography on the representation of receptive profiles by continuous time-frequency signal representations, the question that arises is whether the functional geometry of the visual mechanism directly incorporates a choice of a discrete sampling set suitable for decoding and encoding visual stimuli.

An approach to modelling geometrically the functional architecture of the V_1 visual cortex in terms of contact and sub-Riemannian geometry was developed by Petitot, and by Citti and Sarti, [33, 37, 12]. The purpose of our note here is to highlight some aspects of the contact geometry of the visual cortex, with special attention to a geometric mechanism for the generation of families of Gabor frames. These give rise to a signal analysis setting that is adapted to the underlying contact geometry. We focus here on the specific 3-dimensional case of the manifold of contact elements of a 2-dimensional surface, as this is the setting underlying the model of [37]. We also discuss the case of an associated 5-dimensional contact manifold considered i n [5]. We will not discuss in this paper the more general question of Gabor frames on arbitrary contact manifolds, which we plan to develop elsewhere, since our main goal here is only to investigate some specific geometric aspects of the visual cortex model developed in [37] and in [5].

Replacing the flat planar $\mathbb{R}^2$ as the domain of visual signals with a more general curved Riemann surface S is motivated by the fact that the retina is fixed to the eyeball, hence not flat, that the resolution is not constant (thinner at the center than at the periphery), and that the retinotopic map from the retina to the V_1 cortex along the retino-geniculate-cortical pathway is a conformal map. Thus, the conformal geometry of a Riemann surface is a more suitable model than the flat linear geometry of $\mathbb{R}^2$.

The contact 3-manifold underlying the model of the V_1 cortex of [33, 37] is of the form $M = \mathbb{S}(T^*S)$, namely the unit sphere bundle of the cotangent bundle of a 2-dimensional surface S, also known as the manifold of contact elements of S. One of our main observations here is that the Legendrian circle bundle structure of M, together with the existence of an almost complex structure on the tangent bundle TS, provide a natural choice of a framed lattice (a lattice together with the choice of a basis) on the bundle $\mathcal{E} \oplus \mathcal{E}^\vee$ over the contact 3-manifold manifold M, where $\mathcal{E}$ is the pull-back of TS to M. This lattice determines an associated Gabor system, which has the general form of the Gabor filters considered in [37]. Using the complex analytic method of Bargmann transforms, we investigate when the frame condition is satisfied, so that one obtains Gabor frames for signal analysis consistently associated to the fibers of $\mathcal{E}$.

In terms of the geometric model of the V_1 visual cortex, this shows that the contact geometry directly determines the signal analysis, the Gabor frames property, and the observed shape of the receptive profiles of the V_1 neurons. We show, in particular, that the window function proposed in [37] to model the receptive profiles, together with a scaling of the framed lattice determined by the injectivity radius function of the surface S (representing the retinal surface), give rise to a Gabor system on the bundle of signal planes on the contact 3-manifold M (which models the V_1 cortex) that satisfies the frame condition, hence has optimal signal analysis properties.

On the other hand, the cortical simple cells are organized in hypercolumns, over each point (x, y) of the retina, with respect to their sensitivity on a specific value of a visual feature. These features include orientation, color, spatial frequency, etc. In this context, the hypercolumnar architecture of V_1, for more than one visual feature, is modeled by a fiber bundle of dimension higher than 3 over the retina. Each visual feature considered adds one more dimension to the fibers of the bundle. Thus, for the process of signals from an extended model, which includes more features than the three-dimensional orientation-selectivity framework, it is essential that higher dimensional models have optimal signal analysis properties. In [5], Baspinar et al. extend the orientation selective model to include spacial frequency and phase. However, we show that the lift of the window function, proposed in [37] for the 3-dim model, to the 5-dimensional contact manifold given by the contactization of the symplectization of M, in the form proposed in [5], only defines a Gabor system in a distributional sense, and cannot satisfy the frame condition even distributionally. We show that a simple modification of the proposed window function of [5] restores the desired Gabor frame property and allows for good signal analysis in this higher dimensional model.

2 Signals on manifolds of contact elements

In this section we present the main geometric setting, namely a contact manifold that is either a 3-manifold M given by the manifold of contact elements of a compact 2-dimensional surface, or the 5-manifold given by the contactization of the symplectization of M. These are, respectively, the geometries underlying the models of [34], and of [33, 37], and the model of [5].

The main aspect of the geometry that will play a crucial role in our construction of the associated Gabor frames is the fact that these contact 3-manifolds are endowed with a pair of contact forms α, α_J related through the almost-complex structure J of the tangent bundle TS. They have the property that the circle fibers are Legendrian for both contact forms, while the Reeb vector field of each is Legendrian for the other. This leads to a natural framing, namely a natural choice of a basis for the tangent bundle TM, completely determined by the contact geometry. It consists of the fiber direction ∂_θ and the two Reeb vector fields R_α, R_{α_J}.

2.1 Legendrian circle bundles

The results we discuss in this section apply, slightly more generally, to the case of a 3-manifold M that is a Legendrian circle bundle over a 2 dimensional compact surface S.

The Legendrian condition means that the fiber directions TS^1 inside the tangent bundle TM are contained in the contact planes distribution $\xi \subset TM$. Such Legendrian circle bundles over surfaces are classified, see [27, p. 179]. They are all either given by the unit cosphere bundle $M = \mathbb{S}(T^*S)$, with the contact structure induced by the natural symplectic structure on the cotangent bundle T^*S, or by pull-backs of the contact structure on M to a d-fold cyclic covering $M' \to M$, that exists for d dividing $2g - 2$, where $g = g(S)$ is the genus of S. The case of $M = \mathbb{S}(T^*S)$ is the manifold of contact elements of S. In the following, we will restrict our discussion to this specific case.

In the geometric models of the V_1 cortex developed in [33, 37, 12], the surface S

represents the retinal surface, while the fiber direction in the Legendrian circle bundle $M = \mathbb{S}(T^*S)$ represents an additional orientation variable, which keeps track of how the tangent orientation in TS of a curve in S is lifted to a propagation curve in the visual cortex, where a line is represented by the envelope of its tangents rather than as a set of points.

The fibers of the sphere bundle $\mathbb{S}(T^*S)$ are unit circles S^1, hence they can be seen as parameterizing directions, that is, (oriented) lines in the plane $\mathbb{R}^2 \simeq T^*_{(x,y)}S$. One can also identify the circles with copies of $\mathbb{P}^1(\mathbb{R})$ parameterizing lines in the plane. This would correspond to considering the projectivized cotangent bundle instead of the unit sphere bundle. While these two models are topologically equivalent in dimension $n = 2$, they differ when considering the sub-Riemannian geometry of the rototranslation group $SE(2)$ as model geometry for the neural connectivity of the V_1 cortex, as in [11, 37].

2.2 Liouville tautological 1-form and almost-complex twist

Given a manifold Y, the cotangent bundle T^*Y has a canonical Liouville 1-form, given in coordinates by $\lambda = \sum_i p_i \, dx^i$, or intrinsically as $\lambda_{(x,p)}(v) = p(d\pi(v))$ for $v \in T_xY$ and $\pi : T^*Y \to Y$ the bundle projection. The canonical symplectic form on T^*Y is $\omega = d\lambda$.

Given an almost complex structure J on Y, namely a $(1,1)$ tensor J with $J^2 = -1$, written in coordinates as $J = \sum_{k,\ell} J^k_\ell dx^\ell \otimes \partial_{x_k}$, the twist by J of the tautological Liouville 1-form on T^*Y is given by

$$\lambda_J := \sum_{k,\ell} p_k J^k_\ell dx^\ell,$$

the 2-form $\omega_J = d\lambda_J$ satisfies

$$\omega_J(\cdot,\cdot) = \omega(\hat{J}\cdot,\cdot),$$

where in local coordinates

$$\hat{J} = \begin{pmatrix} J^i_j & 0 \\ \sum_k p_k(\partial_{x_j} J^k_i - \partial_{x_i} J^k_j) & J^j_i \end{pmatrix},$$

see for instance [8].

In particular, in the case of a Riemann surface S, with coordinates $z = x + iy$ on S and $p = (u,v)$ in the cotangent fiber, the tautological 1-form is locally of the form $\lambda = u\,dx + v\,dy$, with $\omega = du \wedge dx + dv \wedge dy$. The twisted tautological form with respect to J given by multiplication by the imaginary unit, $J : (u,v) \mapsto (-v,u)$, given by $\lambda_J = -v\,dx + u\,dy$, with $\omega_J = -dv \wedge dx + du \wedge dy$.

Proposition 2.1. *On the contact 3-manifold $M_w = \mathbb{S}_w(T^*S)$, given by the cosphere bundle of radius w, consider the contact 1-form α induced by the tautological Liouville 1-form λ and the contact 1-form α_J determined by the twisted λ_J. The contact planes of these two contact structures intersect along the circle direction ∂_θ. The Reeb field R_α of α is Legendrian for α_J and the Reeb field R_{α_J} is Legendrian for α. The twist J fixes the ∂_θ generator and exchanges the generators R_{α_J} and R_α.*

Proof. On the contact 3-manifold $M_w = \mathbb{S}_w(T^*S)$, given by the cosphere bundle of radius w, the contact 1-form induced by the tautological Liouville 1-form λ, written in a chart (U,z) on S with local coordinate $z = x + iy$, is given by

$$\alpha = w\cos(\theta)dx + w\sin(\theta)dy, \tag{2.1}$$

where (w,θ) are the polar coordinates in the cotangent fibers, and the corresponding contact planes distribution on $U \times S^1_w$ is generated by the vector fields ∂_θ and $-\sin(\theta)\partial_x +$

$\cos(\theta)\partial_y$, and with the Reeb vector field

$$R_\alpha = w^{-1}\cos(\theta)\partial_x + w^{-1}\sin(\theta)\partial_y \tag{2.2}$$

The contact structure on M_w induced by the twisted Liouville 1-form λ_J is given in the same chart (U, z) by

$$\alpha_J = -w\sin(\theta)dx + w\cos(\theta)dy \tag{2.3}$$

with contact planes spanned by ∂_θ and $\cos(\theta)\partial_x + \sin(\theta)\partial_y$ and with Reeb vector field

$$R_{\alpha_J} = -w^{-1}\sin(\theta)\partial_x + w^{-1}\cos(\theta)\partial_y. \tag{2.4}$$

□

2.3 Symplectization and contactization

Given a contact manifold (M, α), with α a given contact 1-form, one can always form a symplectic manifold $(M \times \mathbb{R}, \omega)$ with $\omega = d(e^s \cdot \alpha)$ with $s \in \mathbb{R}$ the cylinder coordinate. Setting $w = e^s \in \mathbb{R}^*_+$, one has $\omega = dw \wedge \alpha + w\, d\alpha$ on $M \times \mathbb{R}^*_+$. In particular, in the case of the contact manifold $M = \mathbb{S}(T^*S)$ this gives the following.

Lemma 2.2. *The complement of the zero section $T^*S_0 := T^*S \smallsetminus \{0\}$ is the symplectization of the manifold of contact elements $\mathbb{S}(T^*S)$, with symplectic form written in a chart (U, z) of S with $z = x + iy$ in the form*

$$\omega = dw \wedge \alpha + w\, d\alpha = du \wedge dx + dv \wedge dy \tag{2.5}$$

or with the twisted contact and symplectic forms, given in the same local chart by

$$\omega_J = dw \wedge \alpha_J + w\, d\alpha_J = -dv \wedge dx + du \wedge dy. \tag{2.6}$$

for $(u, v) = (w\cos\theta, w\sin\theta)$.

Given a symplectic manifold (Y, ω), if the symplectic form is exact, $\omega = d\lambda$, then one can construct a contactization $(Y \times S^1, \alpha)$ with $\alpha = \lambda - d\phi$, where ϕ is the angle coordinate on S^1. When the symplectic form is not exact, it is possible to construct a contactization if there is some $\hbar > 0$ such that the differential form $\omega/\hbar$ defines an integral cohomology class, $[\omega/\hbar] \in H^2(Y, \mathbb{Z})$. In this case there is a principal $U(1)$-bundle $\mathcal{S}$ on Y with Euler class $e(\mathcal{S}) = [\omega/\hbar]$, endowed with a connection ∇ with curvature $\nabla^2 = \omega/\hbar$. This is also known as the *prequantization* bundle. This connection determines a $U(1)$-invariant 1-form α on $\mathcal{S}$. The non-degeneracy condition for the symplectic form ω implies the contact condition for the 1-form α. Different choices of the potential α of the connection ∇ lead to equivalent contact manifolds up to contactomorphisms, see [18] for a brief summary of symplectization and contactization.

Lemma 2.3. *The contactization of the symplectization of the contact 3-manifold $M = \mathbb{S}(T^*S)$ is the 5-manifold $T^*S_0 \times S^1$ with the contact form*

$$\tilde{\alpha} = \lambda - d\phi = w\alpha - d\phi.$$

Proof. The symplectization of a contact manifold is an exact symplectic manifold, hence it admits a contactization in the simpler form described above. Thus, starting with the contact manifold $M = \mathbb{S}(T^*S)$ for a 2-dimensional compact surface S, endowed with the contact form α as in (2.1) that makes M a Legendrian circle bundle, one obtains the symplectization T^*S_0 with Liouville form $\lambda = w\alpha$, $w = e^s \in \mathbb{R}^*_+$, and the contactization of the resulting exact symplectic manifold $(T^*S_0, \omega = d\lambda)$ is given by $T^*S_0 \times S^1$ with the contact form $\tilde{\alpha} = \lambda - d\phi = w\alpha - d\phi$. □

Remark 2.4. The twist $\alpha \mapsto \alpha_J$ of (2.3) of the contact structure on $M = \mathbb{S}(T^*S)$ induces corresponding twists of the symplectization $\omega \mapsto \omega_J$ as in (2.6) and $\tilde{\alpha} \mapsto \tilde{\alpha}_J = w\alpha_J - d\phi$.

Definition 2.5. *We write*

$$\mathcal{S}(M) := T^*S_0 \quad \text{and} \quad \mathcal{CS}(M) := T^*S_0 \times S^1, \tag{2.7}$$

for the symplectization $\mathcal{S}(M)$ of $M = \mathbb{S}(T^*S)$ and the contactization $\mathcal{CS}(M)$ for this symplectization, endowed with the contact and symplectic forms described above.

In the context of geometric models of the V_1 cortex, the 5-dimensional contact manifold $\mathcal{CS}(M)$ corresponds to the model for the receptive fields considered in [5], where an additional pair of dual variables is introduced, describing phase and velocity of spatial wave propagation.

2.4 The bundle of signal planes

In the model of receptive profiles in the visual cortex (see [33, 37]), signals are regarded as functions on the retinal surface and the receptive profiles are modelled by Gabor filters in these and dual variables. When taking into account the underlying geometric model, however, one needs to distinguish between the local variables (x, y) on a chart $(U, z = x+iy)$ on the surface S (or the local variables (x, y, θ) on the 3-manifold M) and the linear variables in its tangent space $T_{(x,y)}S$. Thus, we think of the retinal signal as a collection of compatible signals in the planes $T_{(x,y)}S$, as (x, y) varies in S. We consider a real 2-plane bundle on the 3-manifold M that describes this geometric space where retinal signals are mapped.

Definition 2.6. *Let $\mathcal{E}$ be the real 2-plane bundle on the contact 3-manifold $M = \mathbb{S}(T^*S)$ obtained by pulling back the tangent bundle TS of the surface S to M along the projection $\pi : \mathbb{S}(T^*S) \to S$ of the unit sphere bundle of T^*S,*

$$\mathcal{E} = \pi^* TS. \tag{2.8}$$

At each point $(x, y, \theta) \in M$, with $z = x + iy$ the coordinate in a local chart (U, z) of S, the fiber $\mathcal{E}_{(x,y,\theta)}$ is the same as the fiber of the tangent bundle $T_{(x,y)}S$. Also let $\mathcal{E}^\vee$ be the dual bundle of $\mathcal{E}$, namely the bundle of linear functional on $\mathcal{E}$,

$$\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E}, \mathbb{R}).$$

Locally the exponential map from TS to S allows for a comparison between the description of signals in terms of the linear variables of TS and the nonlinear variables of S. The linear variables of TS are the ones to which the Gabor filter analysis applies. Thus, in terms of the contact 3-manifold M, we think of a signal as a consistent family of signals on the fibers $\mathcal{E}_{(x,y,\theta)}$, or equivalently a signal on the total space of the 2-plane bundle $\mathcal{E}$. The filters in turn will depend on the dual linear variables of $\mathcal{E}$ and $\mathcal{E}^\vee$. We make this idea more precise in the next subsections.

2.5 Fourier transform relation and signals

Over a compact Riemannian manifold Y, functions on the tangent and cotangent bundles TY and T^*Y are related by Fourier transform in the following way. Let $\mathcal{S}(TY, \mathbb{R})$ denote the vector space of smooth real valued functions on TY that are rapidly decaying along the fiber directions, and similarly for $\mathcal{S}(T^*Y, \mathbb{R})$. Let $\langle \eta, v \rangle_x$ denote the pairing of tangent and cotangent vectors $v \in T_xY$, $\eta \in T_x^*Y$ at a point $x \in Y$. One defines

$$\begin{aligned}
\mathcal{F} :&\mathcal{S}(TY, \mathbb{R}) \to \mathcal{S}(T^*Y, \mathbb{R}) \\
(\mathcal{F}\varphi)_x(\eta) &= \frac{1}{(2\pi)^{\dim Y}} \int_{T_xY} e^{2\pi i \langle \eta, v \rangle_x} \varphi_x(v) \, d\mathrm{vol}_x(v),
\end{aligned}$$

with respect to the volume form on T_xY induced by the Riemannian metric.

Because of this Fourier transform relation, cotangent vectors in T^*Y are sometimes referred to as “spatial frequencies”.

In the model we are considering, the manifold over which signals are defined is the total space $\mathcal{E}$ of the bundle of signal planes introduced in Section 2.4 above, namely real 2-plane bundle $\mathcal{E} = \pi^*TS$. We can easily generalize the setting described above, by replacing the pair of tangent and cotangent bundle TY and T^*Y of a manifold Y with a more general pair of a vector bundle $\mathcal{E}$ and its dual $\mathcal{E}^\vee$. The variables in the fibers of $\mathcal{E}^\vee$ are the spatial frequencies variables of the models of the visual cortex of [33, 37]. In this geometric setting a “signal” is described as follows.

Definition 2.7. *A signal $\mathcal{I}$ is a real valued function on the total space $\mathcal{E}$ of the bundle of signal planes, with $\mathcal{I} \in L^2(\mathcal{E}, \mathbb{R})$, with respect to the measure given by the volume form of M and the norm on the fibers of $\mathcal{E}$ induced by the inner product on TS through the pull-back map. A smooth signal is a smooth function that decays to zero at infinity in the fiber directions, $\mathcal{I} \in \mathcal{C}_0^\infty(\mathcal{E}, \mathbb{R})$.*

The assumption that $\mathcal{I}$ is smooth is quite strong, as one would like to include signals that have sharp contours and discontinuous jumps, but we can assume that such signals are smoothable by convolution with a sufficiently small mollifier function that replaces sharp contours with a steep but smoothly varying gradient.

2.6 Signal analysis and filters

For signals defined over $\mathbb{R}^n$, instead of over a more general manifold, signal analysis is performed through a family of filters (wavelets), and the signal is encoded through the coefficients obtained by integration against the filters. Under good conditions on the family of filters, such as the frame condition for Gabor analysis, both the encoding and the decoding maps are bounded operators, so the signal can be reliably recovered from its encoding through the filters.

For signals on manifolds there is in general no good construction of associated filters for signal analysis, although partial results exist involving splines discretization, diffusive wavelets, or special geometries such as spheres and conformally flat manifolds, see for instance [7, 17, 32]. One of our goals here is to show that geometric modelling of the visual cortex in terms of contact geometry and the description of receptive fields in terms of Gabor frames suggest a general way of performing signal analysis on a specific class of contact manifolds.

The signal analysis model we propose in the following relies on encoding a signal $f : S \to \mathbb{R}$ that is supported on a curved Riemann surface S in terms of a function defined on the total space of the 2-plane bundle $\mathcal{E}$ over the 3-dimensional contact manifold $M = \mathbb{S}(T^*S)$. The restrictions to the fibers of $\mathcal{E}$ provide a collection of signals defined on 2-dimensional linear spaces, which describe the lifts of the original signal $f : S \to \mathbb{R}$ to the local linearizations of S given by the fibers of the tangent bundle TS. The presence of the additional circle coordinate S^1 in the 3-manifold $M = \mathbb{S}(T^*S)$ will account for the fact that the Gabor filters used for signal analysis, which themselves live on the liner fibers of $\mathcal{E}$ include a directional preference specified by the angle coordinate in the fibers of $\mathbb{S}(T^*S)$. In terms of modelling of the visual cortex, what we are presenting in Section 3 below is a functional analytic model of the lifting of signals from the (curved) retinal surface to linear spaces where the Gabor filters corresponding to the receptive fields of the V_1 neurons act to encode the signal. In particular, as we discuss in Section 4 below, we will introduce a version of geometric Bargmann transform. In our setting, since signals are lifted from S to the bundle $\mathcal{E}$, the appropriate Bargmann transform is defined in terms of the duality of the bundles $\mathcal{E}$ and $\mathcal{E}^\vee$ over the contact 3-manifold M.

This version of geometric Bargmann transform differs from other versions previously considered in [2], [3], [16] constructed in terms of the geometry of the Lie algebra of $SE(2)$, or in [4] where frame systems are generated using unitary actions of discrete groups. It also differs from other generalizations of the Bargmann transform such as in [1].

3 Gabor filters on the manifold of contact elements

In this section we present a construction of a family of Gabor systems associated to the contact manifolds described in the previous sections. As above we consider a compact Riemann surface S, and its manifold of contact elements $M = \mathbb{S}(T^*S)$ with the two contact 1-forms α and α_J described in Section 2.2 above.

3.1 Gabor filters and receptive profiles

As argued in [14], simple-cells in the V_1 cortex try to localize at the same time the position and the frequency of a signal, and the shape of simple cells is related to their functionality. However, the uncertainty principle in space-frequency analysis implies that it is not possible to detect, with arbitrary precision, both position and momentum. At the same time, the need for the visual system to process efficiently spatio-temporal information requires optimal extraction and representation of images and their structure. Gabor filters provide such optimality, since they minimize the uncertainty, and are therefore regarded as the most suitable functions to model the shape of the receptive profiles.

The hypothesis that receptive field profiles are Gabor filters is motivated by the analytic properties of Gabor frames. In addition to the minimization of the uncertainty principle mentioned above, the frame condition for Gabor systems provides good encoding and decoding properties in signal analysis, with greater stability to errors than in the case of a Fourier basis. It is therefore a reasonable assumption that such systems would provide an optimal form of signal analysis implementable in biological systems. We will be working here under the hypothesis that receptive field profiles in the V_1 cortex are indeed Gabor filters. In this section we show how to obtain such Gabor filters directly from the contact geometry described in the previous section, while in the next section we discuss the frame condition.

3.2 Gabor systems and Gabor frames

We recall here the notion and basic properties of d-dimensional Gabor systems and Gabor frames, see [24]. Given a point $\lambda = (s, \xi) \in \mathbb{R}^{2d}$, with $s, \xi \in \mathbb{R}^d$, we consider the operator $\rho(\lambda)$ on $L^2(\mathbb{R}^d)$ given by

$$\rho(\lambda) := e^{2\pi i \langle s,\xi\rangle}\, T_s\, M_\xi \tag{3.1}$$

with the translation and modulation operators

$$(T_s f)(t) = f(t - s), \qquad (M_\xi f)(t) = e^{2\pi i\langle \xi,t\rangle} f(t), \tag{3.2}$$

which satisfy the commutation relation

$$T_s M_\xi = e^{-2\pi i\langle s,\xi\rangle} M_\xi T_s.$$

A Gabor system, for a given choice of a "window function" $g \in L^2(\mathbb{R}^d)$ and a $2d$-dimensional lattice $\Lambda = A\mathbb{Z}^{2d} \subset \mathbb{R}^{2d}$, for some $A \in \mathrm{GL}_{2\mathrm{d}}(\mathbb{R})$, consists of the collection of functions

$$\mathcal{G}(g, \Lambda) = \{\rho(\lambda) g\}_{\lambda\in\Lambda}. \tag{3.3}$$

More generally, Gabor systems can be defined in the same way for discrete sets $\Lambda \subset \mathbb{R}^{2d}$ that are not necessarily lattices. We will consider in this paper cases where the discrete set is a translate of a lattice by some vector. In general, one assumes (see [38]) that the discrete set Λ in the construction of the Gabor system is *uniformly discrete*, namely such that

$$q(\Lambda) = \inf\{\|\lambda - \lambda'\| \,|, \lambda, \lambda' \in \Lambda,\ \lambda \neq \lambda'\} > 0\,.$$

This is clearly satisfied in the case where Λ is a translate of a lattice.

A Gabor system $\mathcal{G}(g, \Lambda)$ as in (3.3) is a *Gabor frame* if the functions $\rho(\lambda)g$ satisfy the *frame condition*: there are constants $C, C' > 0$ such that, for all $h \in L^2(\mathbb{R}^d)$,

$$C\,\|h\|^2_{L^2(\mathbb{R}^d)} \leq \sum_\lambda |\langle h, \rho(\lambda)g\rangle|^2 \leq C'\,\|h\|^2_{L^2(\mathbb{R}^d)}. \tag{3.4}$$

The two inequalities in the frame condition ensure that both the encoding map that stores information about signal h into the coefficients $c_\lambda(h) := \langle h, \rho(\lambda)g\rangle$ for $\lambda \in \Lambda$, and the decoding map that reconstructs the signal from these coefficients are bounded linear operators. This ensures good encoding and decoding, even though the Gabor frames $\{\rho(\lambda)g\}$ do not form an orthonormal basis, unlike in Fourier analysis.

Window functions are typically assumed to have a Gaussian shape. It is in general an interesting and highly nontrivial problem of signal analysis to characterize the lattices Λ for which the frame condition (3.4) holds, for a given choice of window function, see [24].

In the modelling of the V_1 cortex, receptive profiles are accurately modelled by Gabor functions, hence it is natural to consider the question of whether there is a lattice Λ, directly determined by the geometric model of V_1, with respect to which the receptive profiles are organized into a Gabor frame system. This is the main question we will be focusing on in the rest of this paper.

3.3 Window function

The construction of Gabor filters we consider here follows closely the model of [37], reformulated in a way that more explicitly reflects the underlying contact geometry described in the previous section. We first show how to obtain the mother function (window function) of the Gabor system and then we will construct the lattice that generates the system of Gabor filters.

Let V and η denote, respectively, the linear variables in the fibers $V \in T_{(x,y)}S \simeq \mathbb{R}^2$, $\eta \in T^*_{(x,y)}S \simeq \mathbb{R}^2$, with $\langle \eta, V\rangle_{(x,y)}$ the duality pairing of $T^*_{(x,y)}S$ and $T_{(x,y)}S$. We write $V = (V_1, V_2)$ and $\eta = (\eta_1, \eta_2)$ in the bases $\{\partial_x, \partial_y\}$ and $\{dx, dy\}$ of the tangent and cotangent bundle determined by the choice of coordinates (x, y) on S.

Definition 3.1. *A window function on the bundle $TS \oplus T^*S$ over S is a smooth real-valued function Φ_0 defined on the total space of $TS \oplus T^*S$, of the form*

$$\Phi_{0,(x,y)}(V, \eta) := \exp\left(-V^t A_{(x,y)} V - i\langle \eta, V\rangle_{(x,y)}\right), \tag{3.5}$$

where A is a smooth section of $T^*S \otimes T^*S$ that is symmetric and positive definite as a quadratic form on the fibers of TS, with the property that at all points (x, y) in each local chart U in S the matrix $A_{(x,y)}$ has eigenvalues uniformly bounded away from zero, $\mathrm{Spec}(\mathrm{A}_{(\mathrm{x},\mathrm{y})}) \subset [\lambda, \infty)$ for some $\lambda > 0$.

Lemma 3.2. *The restriction of a window function Φ_0 as in (3.5) to the bundle $TS \times \mathbb{S}(T^*S)$ determines a real-valued function on the total space of the bundle $\mathcal{E}$, which in a local chart is of the form*

$$\Psi_{0,(x,y,\theta)}(V) := \exp\left(-V^t A_{(x,y)} V - i\langle \eta_\theta, V\rangle_{(x,y)}\right). \tag{3.6}$$

Proof. Consider the restriction of Φ_0 to the bundle $TS \times \mathbb{S}_w(T^*S) \subset TS \oplus T^*S$, for some $w > 0$, over a local chart $(U, z = x + iy)$ of S. This means restricting the variable $\eta \in T^*_{(x,y)}S$ to $\eta = (\eta_1, \eta_2) = (w\cos(\theta), w\sin(\theta))$, with $\theta \in S^1$,

$$\Phi_{0,(x,y)}|_{TS\times \mathbb{S}_w(T^*S)}(V,\theta) = \exp\left(-V^t A_{(x,y)} V - i\langle \eta_\theta, V\rangle_{(x,y)}\right), \tag{3.7}$$

with $\eta_\theta = (w\cos(\theta), w\sin(\theta))$. In particular, we restrict to the case $w = 1$.

We can identify the total space of the bundle $TS \times \mathbb{S}(T^*S)$ with the total space of the bundle of signal planes $\mathcal{E}$ over $M = \mathbb{S}(T^*S)$. Indeed, the direct sum of two vector bundles E_1, E_2 over the same base space S is given by

$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 \,|\, \pi_1(e_1) = \pi_2(e_2)\}.$$

Similarly, when considering sphere bundles

$$E_1 \times \mathbb{S}_w(E_2) = \{(e_1, e_2) \in E_1 \times \mathbb{S}_w(E_2) \,|\, \pi_1(e_1) = \pi_2(e_2)\}.$$

Consider the projection onto the second coordinate, $P : E_1 \oplus E_2 \to E_2$. This projection has fibers $P^{-1}(e_2) = \pi_1^{-1}(\pi_2(e_2))$. Thus, the total space of the bundle $E_1 \oplus E_2$, endowed with the projection P, can be identified with the pull-back $\pi_2^* E_1$ over E_2, with fibers $(\pi_2^* E_1)_{e_2} = \{e_i \in E_1 \,|\, \pi_1(e_1) = \pi_2(e_2)\}$, and similarly when restricting to the sphere bundle of E_2.

Thus, we can write the function in (3.7) equivalently as a real-valued function Ψ_0 on the total space of the bundle $\mathcal{E}$ over the contact 3-manifold M, which is of the form (3.6). $\square$

This provides the reformulation of the Gabor profiles considered in [37] in terms of the underlying geometry of the bundle $\mathcal{E}$ over M.

3.4 Lattices

As above, consider the bundle of signal planes $\mathcal{E}$ over $M = \mathbb{S}(T^*S)$. The two contact forms α and α_J discussed in Section 2.2 determine a choice of basis for TM given by the Legendrian circle fiber direction ∂_θ, together with the two Reeb vector fields R_α and R_{α_J}, each of which is Legendrian for the other contact form. Over a local chart U of S, these two vector fields are given by (2.2), (2.4) and lie everywhere along the TS direction, hence they determine a basis of the fibers $\mathcal{E}_{(x,y,\theta)}$ of the bundle of signal planes for $z = x + iy \in U$.

We denote by $\{R_\alpha^\vee, R_{\alpha_J}^\vee\}$ the dual basis of $\mathcal{E}^\vee$ (over the same chart U of S) characterized by $\langle R_\alpha^\vee, R_\alpha\rangle = 1$, $\langle R_\alpha^\vee, R_{\alpha_J}\rangle = 0$, $\langle R_{\alpha_J}^\vee, R_\alpha\rangle = 0$, $\langle R_{\alpha_J}^\vee, R_{\alpha_J}\rangle = 1$. By the properties of Reeb and Legendrian vector fields, we can identify the dual basis with the contact forms, $\{R_\alpha^\vee, R_{\alpha_J}^\vee\} = \{\alpha, \alpha_J\}$.

Thus, the contact geometry of M determines a canonical choice of a basis $\{R_\alpha, R_{\alpha_J}\}$ for the bundle $\mathcal{E}$ and its dual basis $\{\alpha, \alpha_J\}$ for $\mathcal{E}^\vee$.

This determines bundles of framed lattices (lattices with an assigned basis) over a local chart in M of the form

$$\Lambda_{\alpha,J} := \mathbb{Z}\, R_\alpha + \mathbb{Z}\, R_{\alpha_J} \tag{3.8}$$

$$\Lambda_{\alpha,J}^\vee := \mathbb{Z}\,\alpha + \mathbb{Z}\,\alpha_J\,. \tag{3.9}$$

where $\Lambda_{\alpha,J}$ and $\Lambda_{\alpha,J}^\vee$ here can be regarded as a consistent choice of a lattice $\Lambda_{\alpha,J,(x,y,\theta)}$ (respectively, $\Lambda_{\alpha,J,(x,y,\theta)}^\vee$) in each fiber of $\mathcal{E}$ (respectively, of $\mathcal{E}^\vee$). The bundle of framed lattices

$$\Lambda_{\alpha,J} \oplus \Lambda_{\alpha,J}^\vee \tag{3.10}$$

correspondingly consists of a lattice in each fiber of the bundle $\mathcal{E} \oplus \mathcal{E}^\vee$ over M. We will also equivalently write the bundle of lattices (3.10) in the form $\Lambda + \Lambda_J$ with

$$\Lambda = \mathbb{Z}\, R_\alpha \oplus \mathbb{Z}\, \alpha, \quad \Lambda_J = \mathbb{Z}\, R_{\alpha_J} \oplus \mathbb{Z}\, \alpha_J \,. \tag{3.11}$$

In the following, we will often simply use the term "lattice" to indicate bundles of framed lattices over M as above.

Lemma 3.3. *The choice of the window function Ψ_0 described in Section 3.3, together with the lattice (3.10), determine a Gabor system*

$$\mathcal{G}(\Psi_0, \Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$$

which consists, at each point $(x, y, \theta) \in M$ of the Gabor system

$$\mathcal{G}(\Psi_{0,(x,y,\theta)}, \Lambda_{\alpha,J,(x,y,\theta)} \oplus \Lambda^\vee_{\alpha,J,(x,y,\theta)})$$

in the space $L^2(\mathcal{E}_{(x,y,\theta)})$.

Proof. The Gabor functions in $\mathcal{G}(\Psi_0, \Lambda + \Lambda_J)$ are of the form

$$\rho(\lambda)\Psi_0 = \rho(\xi)\rho(W)\,\Psi_0 = e^{2\pi i\langle \xi, V\rangle}\,\Psi_0(V - W),$$

for $\lambda = (\xi, W)$ with $\xi \in \Lambda^\vee_{\alpha,J} \subset \mathcal{E}^\vee$ and $W \in \Lambda_{\alpha,J} \subset \mathcal{E}$. □

3.5 Injectivity radius function and lattice truncation

In order to adapt this construction to a realistic model of signal processing in the V_1 cortex, one needs to keep into account the fact that in reality only a finite, although large, number of Gabor filters in the collection $\mathcal{G}(\Psi_0, \Lambda + \Lambda_J)$ contribute to the analysis of the retinal signals. This number is empirically determined by the structure of the neurons in the V_1 cortex. This means that there is some (large) cut-off size $R_{\max} > 0$ such that the part of the lattice that contributes to the available Gabor filters is contained in a ball of radius $R_{\max}$.

There is also an additional constraint that comes from the geometry. Namely, we are using Gabor analysis in the signal planes determined by the vector bundle $\mathcal{E}$ to analyze a signal that is originally stored on the retinal surface S. Lifting the signal from S to the fibers of $\mathcal{E}$ and consistency or results across nearby fibers is achieved through the exponential map

$$\exp_{(x,y)} : T_{(x,y)}S \to S$$

from the tangent bundle of S (of which $\mathcal{E}$ is the pull-back to M) to the surface. At a given point $(x, y) \in S$ let $R_{inj}(x, y) > 0$ be the supremum of all the radii $R > 0$ such that the exponential map $\exp_{(x,y)}$ is a diffeomorphism on the ball $B(0, R)$ of radius R in $T_{(x,y)}S$. For a compact surface S, we obtain a continuous *injectivity radius function* given by $R_{inj} : S \to \mathbb{R}^*_+$ given by $(x, y) \mapsto R_{inj}(x, y)$.

Thus, to obtain good signal representations and signal analysis in the signal planes, we want that the finitely many available lattice points that perform the shift operators $T_W = \rho(W)$ in the Gabor system construction lie within a ball of radius R_{inj} in the fibers of $\mathcal{E}$.

It is reasonable to assume that the maximal size $R_{\max}$, determined by empirical data on neurons in the visual cortex, will be in general very large, and in particular larger than the maximum over the compact surface S of the injectivity radius function. This means that, in order to match these two bounds, we need to consider a scaled copy of the lattice $\Lambda_{\alpha,J}$. We obtain the following scaling function.

Lemma 3.4. *Let $b_M : M \to \mathbb{R}^*_+$ be the function given by*

$$b_M(x, y, \theta) := \frac{R_{inj}(x, y)}{R_{\max}}, \tag{3.12}$$

where $R_{inj}(x, y)$ is the injectivity radius function and $R_{\max} > 0$ is an assigned constant. For $R_{\max} > \max_{(x,y)\in S} R_{inj}(x, y)$, consider the rescaled lattice

$$\Lambda_{b,\alpha,J} := b_M \Lambda_{\alpha,J} = \mathbb{Z}\, b_M\, R_\alpha + \mathbb{Z}\, b_M\, R_{\alpha_J} \quad \text{and} \quad \left\{ \begin{array}{l} \Lambda_b = \mathbb{Z}\, b_M\, R_\alpha \oplus \mathbb{Z}\, \alpha, \\ \Lambda_{b,J} = \mathbb{Z}\, b_M\, R_{\alpha_J} \oplus \mathbb{Z}\, \alpha_J \end{array} \right. . \tag{3.13}$$

All the lattice points of the original lattice $\Lambda_{\alpha,J}$ that are within the ball of radius $R_{\max}$ correspond to lattice points of the rescaled $\Lambda_{b,\alpha,J}$ that are within the ball of radius $R_{inj}(x, y)$ in $\mathcal{E}_{(x,y,\theta)}$. In particular, for B a ball of measure 1 in $\mathcal{E}_{(x,y,\theta)}$, and $N(r) = \#\{\lambda \in \Lambda_{b,\alpha,J} \cap r \cdot B\}$, we have

$$D^-(\Lambda_{b,\alpha,J}) = \liminf_{r\to\infty} \frac{N(r)}{r} = b_M^{-1} > 1\,. \tag{3.14}$$

Proof. The first statement is clear by construction. Moreover, under the assumption that $R_{\max} > \max_{(x,y)\in S} R_{inj}(x, y)$, the function b_M of (3.12) is everywhere smaller than one,

$$b_M(x, y, \theta) < 1, \quad \forall (x, y, \theta) \in M\,, \tag{3.15}$$

so that the density $D^-(\Lambda_{b,\alpha,J}) > 1$. ∎

Remark 3.5. Note that we only need to rescale the $\Lambda_{\alpha,J}$ part of the lattice in $\mathcal{E}$ and not the $\Lambda^\vee_{\alpha,J}$ part of the lattice in $\mathcal{E}^\vee$, since the $\Lambda^\vee_{\alpha,J}$ part only contributes modulation operators M_ξ that do not move the coordinates outside of the injectivity ball of the exponential map, unlike the translation operators T_W with $W \in \Lambda_{\alpha,J}$.

We can also make the choice here to scale both parts of the lattice by the same factor $b = b_M$, and work with the scaled lattice $\Lambda_{b,\alpha,J} \oplus \Lambda^\vee_{b,\alpha,J}$ even if the scaling of the modulation part is not necessary by the observation of Remark 3.5 above. The difference between these two choices can be understood geometrically in the following way. One usually normalizes the choice of the Reeb vector field of a contact form by the requirement that the pairing is $\langle \alpha, R_\alpha \rangle = 1 = \langle \alpha_J, R_{\alpha,J} \rangle$. However, one can make a different choice of normalization. Scaling only the $\Lambda_{\alpha,J}$ part of the lattice and not the $\Lambda^\vee_{\alpha,J}$ corresponds to changing this normalization, while scaling both parts means that one maintains the normalization. As will be clear in the argument of Proposition 4.13, these two choices are in fact equivalent and give the same signal analysis properties.

4 The Gabor frame condition

In this section we check that the Gabor systems introduced above on the bundle of signal spaces $\mathcal{E}$ satisfy the frame condition. This condition is necessary for discrete systems of Gabor filters to perform good signal analysis, in the sense that signals can be reconstructed from their measurements by the filters. In the usual setting of Gabor systems with Gaussian window on a single vector space $\mathbb{R}^n$, the frame condition has been extensively studied. However, while in the 1-dimensional case the frame condition can be characterized in terms of a density property for the lattice ([28, 38]), in higher dimensions the question of whether a Gabor frame with Gaussian window in $\mathbb{R}^n$ and a given lattice $\Lambda \subset \mathbb{R}^{2n}$ satisfies the frame condition is generally open and very difficult to assess, see [24]. Since we are specifically interested here in the 2-dimensional case, we will follow the method developed in [24], based on the Bargmann transform, adapted to our geometric setting.

We discuss separately the case where, in a local chart U in S, the quadratic form A in the window function Ψ_0 is diagonal in the basis $\{R_\alpha, R_{\alpha_J}\}$ and the general case where it is not diagonal. The first case has the advantage that it reduces to one-dimensional Gabor systems, for which we can reduce the discussion to a famous result of Lyubarskiĭ and Seip, [28, 38], after the slightly different form of the window function is accounted for. The more general case can be dealt with along the lines of the results of [24] for 2-dimensional Gabor systems. In particular, the analysis of the frame condition relies on the complex analytic technique of Bargmann transform and sampling.

As discussed in Section 2.6 above, the notion of geometric Bargmann transform that we introduce here, for the purpose of investigating the frame condition, is defined in terms of the geometry of the dual pair of vector bundles $\mathcal{E}$ and $\mathcal{E}^\vee$ over the contact 3-manifold $M = \mathbb{S}(T^*S)$, since in our setting retinal signals $f : S \to \mathbb{R}$ are lifted to signals that live on the linear fibers $\mathcal{E}$, with the angular coordinate of the circle fibers of $\mathbb{S}(T^*S)$ accounting for the directionality of the Gabor filters.

4.1 Gabor frame condition

Let $\mathcal{E}$ be the bundle of signal planes on the contact 3-manifold M as above. Let Ψ_0 be a window function, which we assume of the form (3.6). Suppose given a lattice bundle Λ, namely a bundle over M with fiber isomorphic to $\mathbb{Z}^4$, where the fiber $\Lambda_{(x,y,\theta)}$ is a lattice in $(\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)}$. We form the Gabor system $\mathcal{G}(\Psi_0, \Lambda)$ as in Lemma 3.3, with Gabor functions $\rho(\lambda_{(x,y,\theta)})\Psi_0|_{\mathcal{E}_{(x,y,\theta)}}$, with $\lambda_{(x,y,\theta)} \in \Lambda_{(x,y,\theta)}$.

Definition 4.1. *The Gabor system $\mathcal{G}(\Psi_0, \Lambda)$ satisfies the smooth Gabor frame condition on M if there are smooth $\mathbb{R}^*_+$-valued functions C, C' on the local charts of M, such that the frame condition holds pointwise in (x, y, θ),*

$$C_{(x,y,\theta)} \, \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})} \leq \sum_{\lambda_{(x,y,\theta)} \in \Lambda_{(x,y,\theta)}} |\langle f, \rho(\lambda_{(x,y,\theta)})\Psi_0\rangle|^2 \leq C'_{(x,y,\theta)} \, \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})} . \quad (4.1)$$

Note that, although, the manifold M is compact, so that globally defined continuous functions $C, C' : M \to \mathbb{R}_+$ would have a minimum and a maximum that are strictly positive and finite, in the condition above we are only requiring that the functions C, C' are defined on the local charts, without necessarily extending globally to M. Indeed, since global vector fields on an orientable compact surface S necessarily have singularities (unless $S = T^2$), the frame condition will not in general extend globally, while it holds locally within each chart, with not necessarily uniformly bounded C, C'. If these functions extend globally to M, then a stronger global frame condition

$$C_{\min} \, \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})} \leq \sum_{\lambda_{(x,y,\theta)} \in \Lambda_{(x,y,\theta)}} |\langle f, \rho(\lambda_{(x,y,\theta)})\Psi_0\rangle|^2 \leq C'_{\max} \, \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})}$$

would also be satisfied, but one does not expect this to be the case, except in special cases like the parallelizable $S = T^2$. In the case directly relevant to the modeling of the primary visual cortex, one assumes that the retinal surface is represented by a chart $U \subset S$ with $S = S^2$ a sphere.

4.2 The diagonal case: dimensional reduction

Consider first the case where the quadratic form A in (3.6) is diagonal in the basis $\{R_\alpha, R_{\alpha_J}\}$ of the bundle $\mathcal{E}$.

First observe that, in a local chart U of S, the unit vector $\eta_\theta \in T^*S$ is in fact the vector $\eta_\theta = (\cos(\theta), \sin(\theta))$ in the basis $\{dx, dy\}$, which is the dual basis element α, as in (2.1). Thus, the window function (3.6) used in [37] is of the form

$$\Psi_{0,(x,y,\theta)}(V) = \rho(\frac{1}{2\pi}(0,1))\hat{\Psi}_{0,(x,y,\theta)}(V) \quad (4.2)$$

where

$$\hat{\Psi}_{0,(x,y,\theta)}(V) := \exp\left(-V^t A_{(x,y)} V\right) \tag{4.3}$$

and $(0,-1) \in \Lambda$ is the covector $-\eta_\theta(x,y) = \alpha|_{(x,y,\theta)}$. Thus, the Gabor system can be equivalently described as

$$\mathcal{G}(\Psi_0, \Lambda + \Lambda_J) = \mathcal{G}(\hat{\Psi}_0, \hat{\Lambda} + \Lambda_J)$$

$$\hat{\Lambda} = \xi_0 + \Lambda = \{(W,\xi) \in \mathcal{E} \oplus \mathcal{E}^\vee \,|\, W \in \mathbb{Z}R_\alpha,\ \xi \in \xi_0 + \mathbb{Z}\alpha\} \quad \text{with } \xi_0 := -\frac{1}{2\pi}\alpha \in \mathcal{E}^\vee\,. \tag{4.4}$$

Note that $\hat{\Lambda}$ is no longer a lattice (a discrete abelian *subgroup* in each fiber $\mathcal{E}_{(x,y,\theta)} \oplus \mathcal{E}^\vee_{(x,y,\theta)}$ in the local chart): it is however a uniformly discrete set given by the translate $\xi_0 + \Lambda$.

Lemma 4.2. *If the quadratic form A in (3.6) is diagonal, $A = \mathrm{diag}(\kappa_1^2, \kappa_2^2)$, in the basis $\{R_\alpha, R_{\alpha_J}\}$ of $\mathcal{E}$ in a local chart, then the Gabor frame condition for $\mathcal{G}(\Psi_0, \Lambda + \Lambda_J)$ is equivalent to the frame condition for two uncoupled problems for the one-dimensional Gabor systems $\mathcal{G}(\psi_0, \Lambda)$ and $\mathcal{G}(\phi_0, \Lambda_J)$, with $\psi_0(V_1) = \exp(-\kappa_1^2 V_1^2 - iV_1)$ and $\phi_0(V_2) = \exp(-\kappa_2^2 V_2^2)$.*

Proof. Given the duality pairing relations between the contact forms α, α_J and their Reeb vector fields R_α and R_{α_J}, if we write the vectors $V \in \mathcal{E}_{(x,y,,\theta)}$ in coordinates $V = V_1\, R_\alpha + V_2\, R_{\alpha_J}$ over the local chart, then the window function is written in the form

$$\Psi_{0,(x,y,\theta)}(V_1, V_2) = \exp(-\kappa_1^2 V_1^2 - iV_1) \cdot \exp(-\kappa_2^2 V_2^2) = \psi_0(V_1) \cdot \phi_0(V_2),$$

and the Gabor system is of the form

$$(\rho(\lambda)\Psi_0)(V) = (\rho(\lambda_1)\psi_0)(V_1) \cdot (\rho(\lambda_2)\phi_0)(V_2)$$
$$\lambda_1 = (\xi_1, W_1) \in \Lambda \quad \text{and} \quad \lambda_2 = (\xi_2, W_2) \in \Lambda_J\,.$$

This means that, in this case, the Gabor frame condition problem for $\mathcal{G}(\Psi_0, \Lambda + \Lambda_J)$ reduces to two uncoupled problems for the one-dimensional Gabor systems $\mathcal{G}(\psi_0, \Lambda)$ and $\mathcal{G}(\phi_0, \Lambda_J)$. The frame condition for $\mathcal{G}(\Psi_0, \Lambda + \Lambda_J)$ is satisfied iff it is satisfied for $\mathcal{G}(\psi_0, \Lambda)$ and $\mathcal{G}(\phi_0, \Lambda_J)$, where the first problem, by the discussion above, is equivalent to the frame condition for the system $\mathcal{G}(\hat{\psi}_0, \hat{\Lambda})$ with $\hat{\Lambda} = \xi_0 + \Lambda$ and $\hat{\psi}_0(V_1) = \exp(-\kappa_1^2 V_1^2)$. ☐

Proposition 4.3. *The functions in the Gabor system $\mathcal{G}(\Psi_0, \Lambda + \Lambda_J)$ are* not *frames.*

Proof. The second case above is a one-dimensional Gabor system with a Gaussian window function $g(t) = e^{-\kappa^2 t^2}$ and the lattice $\mathbb{Z}^2$, while the first case is a one-dimensional Gabor system with a modified window function of the form $g(t) = e^{-\kappa^2 t^2 - iat}$ and the lattice $\mathbb{Z}^2$ or equivalently a window function $\hat{g}(t) = e^{-\kappa^2 t^2}$ and the discrete set $(0,a) + \mathbb{Z}^2$.

For a lattice $\Lambda = A\mathbb{Z}^d$ with $A \in \mathrm{GL_d}(\mathbb{R})$ the density is given by $s(\Lambda) = |\det(A)|$. In particular it is $s(\Lambda) = 1$ for the standard lattice $\mathbb{Z}^2$. The *density theorem* for Gabor frames, [26] (see also Proposition 2 of [24]), states that if a Gabor system $\mathcal{G}(g,\Lambda)$ is a frame in $L^2(\mathbb{R}^d)$ and the window is a rapid decay function $g \in \mathcal{S}(\mathbb{R}^d)$, then necessarily $s(\Lambda) < 1$. Thus, these one-dimensional Gabor systems are not frames, hence the original system $\mathcal{G}(\Psi_0, \Lambda + \Lambda_J)$ also does not satisfy the frame condition. ☐

On the other hand, the situation changes when one takes into account the scaling of the lattice discussed in Section 3.5.

Proposition 4.4. *Consider the rescaled lattices $\Lambda_{b,\alpha,J}$, Λ_b, $\Lambda_{b,J}$ of (3.13). The system $\mathcal{G}(\Psi_0, \Lambda_b + \Lambda_{b,J})$ does satisfy the frame condition.*

Proof. The Gabor frame question for the system $\mathcal{G}(\Psi_0, \Lambda_b + \Lambda_{b,J})$ reduces to the question of whether the one-dimensional systems $\mathcal{G}(\phi_0, \Lambda_{b,J})$ and $\mathcal{G}(\hat{\psi}_0, \hat{\Lambda}_b)$ with $\hat{\Lambda}_b = \xi_0 + \Lambda_b$ are frames.

In the case of one-dimensional systems, there is a complete characterization of when the frame condition is satisfied, [28, 38, 39]. This characterization is obtained by reformulating the problem in terms of a complex analysis problem of sampling and interpolation in Bargmann-Fock spaces. In the case of a Gaussian window function ψ and a uniformly discrete set $\Lambda \subset \mathbb{R}^2$, it is proved in [38] that the Gabor system $\mathcal{G}(\psi, \Lambda)$ is a frame if and only if the lower Beurling density satisfies $D^-(\Lambda) > 1$, where

$$D^-(\Lambda) = \lim_{r\to\infty} \inf \frac{N_\Lambda^-(r)}{r^2},$$

with $N_\Lambda^-(r)$ the smallest number of points of Λ contained in a scaled copy $r\mathcal{I}$ of a given set $\mathcal{I} \subset \mathbb{R}^2$ of measure one, with measure zero boundary. The value $D^-(\Lambda)$ is independent of the choice of the set $\mathcal{I}$. In the case of a rank two lattice this corresponds to the condition $s(\Lambda) < 1$, which is therefore also sufficient.

Thus, the one-dimensional systems $\mathcal{G}(\phi_0, \Lambda_{b,J})$ and $\mathcal{G}(\hat{\psi}_0, \hat{\Lambda}_b)$ are frames if and only if $s(\Lambda_{b,J}) < 1$ and $s(\Lambda_b) < 1$, since the translate $\hat{\Lambda}_b$ and Λ_b have the same lower Beurling density. Since the scaling function satisfies $b_M < 1$ everywhere on M, as in (3.15), we have seen in Lemma 3.4 that these conditions are satisfied. It follows that the Gabor system $\mathcal{G}(\Psi_0, \Lambda_b + \Lambda_{b,J})$ is a frame. □

4.3 The non-diagonal case: Bargmann transform

In the more general case where the quadratic form in Ψ_0 is not necessarily diagonal in the basis $\{R_\alpha, R_{\alpha,J}\}$ in a local chart, the question of whether the Gabor system $\mathcal{G}(\Psi_0, \Lambda_b + \Lambda_{b,J})$ satisfies the frame condition can still be reformulated in terms of sampling and interpolation in Bargmann-Fock spaces, see [24].

4.3.1 Bargmann transform and Gabor frames

The Bargmann transform of a function f in $L^2(\mathbb{R}^n)$ is defined as

$$\mathcal{B}f(z) = \int_{\mathbb{R}^n} f(t) e^{2\pi t\cdot z - \pi t^2 - \frac{\pi}{2} z^2}\, dt\,, \tag{4.5}$$

where, for $z \in \mathbb{C}^n$ we write $z = x + iw$ for some $x, w \in \mathbb{R}^n$ and $z^2 = (x + iw) \cdot (x + iw) = x \cdot x - w \cdot w + i2x \cdot w$ and $|z|^2 = x^2 + w^2$. It is a unitary transformation from $L^2(\mathbb{R}^n)$ to the Bargmann-Fock space $\mathcal{F}_n^2$, which consists of entire functions of $z \in \mathbb{C}^n$ with finite norm

$$\|F\|^2_{\mathcal{F}_n^2} = \int |F(z)|^2\, e^{-\pi|z|^2}\, dz\, < \infty\,, \tag{4.6}$$

induced by the inner product

$$\langle F, G\rangle_{\mathcal{F}_n^2} = \int_{\mathbb{C}^n} F(z)\, \overline{G(z)}\, e^{-\pi|z|^2}\, dz\,.$$

We also consider the Bargmann-Fock space $\mathcal{F}_n^\infty$, which is the space of entire functions on $\mathbb{C}^n$ with

$$\|F\|^2_{\mathcal{F}_n^\infty} = \sup_{z\in\mathbb{C}^n} |F(z)|\, e^{-\frac{\pi|z|^2}{2}}\, < \infty\,. \tag{4.7}$$

There is a well known relation between the Bargmann transform and Gabor systems with Gaussian window function, see for instance [22, 24]. In our setting, because of the form (4.2) of the window function, we need a simple variant of this relation between Gabor systems and Bargmann transform which we now illustrate.

A set $\Lambda \subset \mathbb{C}^n$ is a *sampling set* for $\mathcal{F}^2_n$ if there are constants $C, C' > 0$, such that, for all $F \in \mathcal{F}^2_n$,

$$C \cdot \|F\|^2_{\mathcal{F}^2_n} \leq \sum_{\lambda \in \Lambda} |F(\lambda)|^2 e^{-\pi|\lambda|^2} \leq C' \cdot \|F\|^2_{\mathcal{F}^2_n} .$$

A set $\Lambda \subset \mathbb{C}^n$ is a *set of uniqueness* for $\mathcal{F}^\infty_n$ if a function $F \in \mathcal{F}^\infty_n$ satisfying $F(\lambda) = 0$ for all $\lambda \in \Lambda$ must vanish identically, $F \equiv 0$. For $\Lambda \subset \mathbb{C}^n$, let $\bar{\Lambda} = \{\bar{\lambda} \,|\, \lambda \in \Lambda\}$.

We consider as in [23] the modulation spaces $M^p(\mathbb{R}^n)$ as the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ with Gabor transform with bounded L^p norm, $\|V_\varphi f\|_p < \infty$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, where

$$V_\varphi f = \langle f, M_\xi T_x \varphi\rangle = \int_{\mathbb{R}^d} f(t)\overline{\varphi(t-x)} e^{-2\pi i \xi \cdot t}\, dt .$$

Similarly, the modulation space $M^\infty(\mathbb{R}^n)$ is the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|V_\varphi f\|_\infty < \infty$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Proposition 4.5. *Let $\Lambda \subset \mathbb{C}^n$ be a lattice and let $\phi(x) = e^{-\pi\,|x|^2} e^{-2\pi i\, a \cdot x} \in L^2(\mathbb{R}^n)$, for some fixed $a \in \mathbb{R}^n$. Then the following conditions are equivalent.*

1. *The Gabor system $\mathcal{G}(\phi, \Lambda)$ is a frame.*
2. *The set $\bar{\Lambda}_a := \bar{\Lambda} + ia$ is a sampling set for $\mathcal{F}^2_n$.*
3. *The set $\bar{\Lambda}_a$ is a set of uniqueness for $\mathcal{F}^\infty_n$.*

Proof. For the proof of $1 \iff 2$ it suffices to prove that

$$|\langle f, M_w T_x \phi\rangle| = |\mathcal{B}(x - i(w+a))| e^{-\frac{\pi|(x-i(w+a))|^2}{2}} .$$

We have

$$\begin{aligned} V_\phi f(x,w) &= \int_{\mathbb{R}^n} f(t) e^{-\pi(t-x)^2} e^{-2\pi i (a\cdot(t-x))} e^{-2\pi i (w\cdot t)} dt \\ &= e^{2\pi i (a\cdot x)} \int_{\mathbb{R}^n} f(t) e^{-\pi t^2 + 2\pi t x - \pi x^2} e^{-2\pi i (a+w)\cdot t} dt \\ &= e^{2\pi i a\cdot x} e^{-\pi i x\cdot(a+w)} e^{-\frac{\pi}{2}(x^2+(a+w)^2)} \int_{\mathbb{R}^n} f(t) e^{-\pi t^2} e^{2\pi t\cdot(x-i(w+a))} e^{-\frac{\pi}{2}(x-i(a+w))^2} dt . \end{aligned}$$

Moreover, for $z' = x + i(w+a)$,

$$V_\phi f(x,w) = e^{-\frac{\pi}{2}|z'|^2} e^{-\pi i x\cdot \Im(z')} e^{2\pi i (a\cdot x)} \mathcal{B}f(\overline{z'})$$

Thus, $|V_\phi f(x,w)| = |\mathcal{B}\, f(\overline{z'})| e^{-\frac{\pi}{2}|z'|^2} = |\mathcal{B}\, f(x - i(w+a))| e^{-\frac{\pi|(x-i(w+a))|^2}{2}}$. Thus, we obtain

$$\sum_{\lambda\in\Lambda} |V_\phi f(\lambda)| = \sum_{z'\in\bar{\Lambda}_a} |\mathcal{B}\, f(z')| e^{-\frac{\pi}{2}|z'|^2} ,$$

and $\sum_{\lambda\in\Lambda} |V_\phi f(\lambda)| \asymp \|f\|_{L^2(\mathbb{R}^n)}$ if and only if

$$\sum_{z'\in\bar{\Lambda}_a} |\mathcal{B}\, f(z')| e^{-\frac{\pi}{2}|z'|^2} \asymp \|\mathcal{B}\, f\|_{\mathcal{F}^2_n} .$$

To prove $2 \iff 3$, starting with the assumption that $\bar{\Lambda}_a$ is a set of sampling for $\mathcal{F}^2_n$, let $F \in \mathcal{F}^\infty_n$ be such that $F(\lambda) = 0$ for all $\lambda \in \bar{\Lambda}_a$. The Bargmann-Fock space $\mathcal{F}^\infty_n$ is related to the modulation space $M^\infty(\mathbb{R}^n)$ through the Bargmann transform (4.5),

$$\mathcal{F}^\infty_n = \mathcal{B}(M^\infty(\mathbb{R}^n)) .$$

Thus, there exists an element $f \in M^\infty(\mathbb{R}^n)$ such that $\mathcal{B} f = F$. Thus, we have $\mathcal{B} f(\lambda) = 0$, for all $\lambda \in \bar{\Lambda}_a$, hence $\langle f, \pi(\lambda)\phi\rangle = 0$, for all $\lambda \in \Lambda$. The equivalence $1 \iff 2$ then implies that $f \equiv 0$, hence $F \equiv 0$.

Conversely, suppose that $\bar{\Lambda}_a$ is a set of uniqueness for $\mathcal{F}_n^\infty$. Theorem 3.1 of [23] shows that the frame condition for the Gabor system $\mathcal{G}(\phi, \Lambda)$, for a window $\phi \in \mathcal{S}(\mathbb{R}^n)$, is equivalent to the condition that the Gabor transform map is one-to-one as a map

$$V_\phi : M^\infty(\mathbb{R}^n) \to \ell^\infty(\Lambda)\,, \quad V_\phi : f \mapsto V_\phi f|_\Lambda\,. \tag{4.8}$$

Since we have $\phi \in \mathcal{S}(\mathbb{R}^n)$, it suffices to prove that the Gabor transform $f \mapsto V_\phi f|_{\bar{\Lambda}_a}$ is one-to-one as a map $M^\infty(\mathbb{R}^n) \to \ell^\infty(\bar{\Lambda}_a)$.

Let D denote the map $D : M^\infty(\mathbb{R}^n) \to \ell^\infty(\Lambda)$ given by

$$D : f \mapsto \{\mathcal{B} f(\lambda)\}_{\lambda\in\Lambda}\,,$$

and let $T : \ell^\infty(\Lambda) \to \ell^\infty(\Lambda_a)$ be given by

$$T : \{c_\lambda\}_{\lambda\in\Lambda} \mapsto \{e^{\pi i\lambda_1(\lambda_2+a)}e^{-|\lambda+(0,a)|^2/2}c_\lambda\}_{\lambda+(0,a)\in\Lambda_a}\,,$$

The operator V_ϕ of (4.8) is the composite $V_\phi = T \circ D$, which is injective since both T and D are. ☐

Remark 4.6. In particular this shows that, with the window functions $\tilde{\phi}(x) = e^{-\pi x^2}$ and $\phi(x) = e^{-\pi x^2}e^{-2\pi i(a\cdot x)}$, the Gabor system $\mathcal{G}(\phi, \Lambda)$ is a frame if and only if $\mathcal{G}(\tilde{\phi}, \Lambda)$ is a frame.

Indeed, for the window $\tilde{\phi}$ the system $\mathcal{G}(\tilde{\phi}, \Lambda)$ is a frame iff the system $\mathcal{G}(\tilde{\phi}, \Lambda_a)$ is a frame and the latter is equivalent to

$$\sum_{z\in\bar{\Lambda}} |\mathcal{B} f(z - ia)|e^{-\frac{\pi}{2}|z|^2} \asymp ||\mathcal{B} f||_{\mathcal{F}_n^2}\,,$$

which we have seen is equivalent to $\mathcal{G}(\phi, \Lambda)$ being a frame.

4.3.2 Geometric Bargmann transform

We apply this Bargmann transform argument to our geometric setting. The bundle $\mathcal{E}$ is endowed with an almost complex structure, coming from the identification $\mathcal{E} = \pi^* TS$ with S a Riemann surface, hence the dual $\mathcal{E}^\vee$ can also be endowed with an almost complex structure. However, for the purpose of applying the Bargmann transform argument in our setting, we just need to consider the bundle $\mathcal{E} \oplus \mathcal{E}^\vee$ as a complex 2-plane bundle over M. First note that the local bases $\{R_\alpha, R_{\alpha_J}\}$ of $\mathcal{E}$ and $\{\alpha, \alpha_J\}$ of $\mathcal{E}^\vee$ determine a local isomorphism between $\mathcal{E}$ and $\mathcal{E}^\vee$. For $(W, \eta) \in (\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)}$, with $W = W_1 R_\alpha + W_2 R_{\alpha_J}$ and $\eta = \eta_1\alpha + \eta_2\alpha_J$, we define $J : \mathcal{E} \oplus \mathcal{E}^\vee \to \mathcal{E} \oplus \mathcal{E}^\vee$ with $J^2 = -1$ by setting

$$J\,(W, \eta) := (\eta, -W) = \eta_1\, R_\alpha + \eta_2\, R_{\alpha_J} - W_1\,\alpha - W_2\,\alpha_J\,.$$

We can then take $W + i\eta := (W, \eta)$ with scalar multiplication by $\lambda \in \mathbb{C}$, $\lambda = x + iy$ with $x, y \in \mathbb{R}$ given by $\lambda \cdot (W + i\eta) = (x + y\,J)\,(W, \eta)$. This gives a fiberwise identification

$$\mathcal{I} : (\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)} \stackrel{\simeq}{\to} \mathbb{C}^2 \quad (W, \eta) \mapsto z = (z_1, z_2) = (W_1 + i\eta_1, W_2 + i\eta_2)\,. \tag{4.9}$$

Given the choice of a window function $\Psi_{0,(x,y,\theta)}(V)$ as in (3.6), with a quadratic form on the fibers of $\mathcal{E}$ over the local chart, determined by a smooth section A of $T^*S \otimes T^*S$ that is symmetric and positive definite, we consider an associated quadratic form

$$\mathcal{Q} : \mathcal{E} \oplus \mathcal{E}^\vee \to \mathbb{C}, \quad \mathcal{Q}_{(x,y,\theta)}(W + i\eta) := W^t\, A_{(x,y)}\, W + 2i\langle\eta, W\rangle_{(x,y,\theta)} - \eta^t\,\eta, \tag{4.10}$$

where $\langle \eta, W\rangle$ is the duality pairing of $\mathcal{E}$ and $\mathcal{E}^\vee$, and $\eta^t\,\eta$ denotes the pairing with respect to the metric in $\mathcal{E}^\vee$ determined by the metric on S. We use the notation

$$\mathcal{Q}(z) := \mathcal{Q}\circ\mathcal{I}^{-1}(z) \quad \text{and} \quad V \bullet z := V^t \frac{A_{(x,y)}}{2\pi} W + i\langle \eta, V\rangle\,. \tag{4.11}$$

We also define $\tilde{\mathcal{Q}} : \mathcal{E}\oplus\mathcal{E}^\vee \to \mathbb{C}$ as

$$\tilde{\mathcal{Q}}_{(x,y,\theta)}(W,\eta) := \frac{\pi}{2}\big(W^t \frac{A_{(x,y)}}{\pi} W + (\eta + \frac{\eta_\theta}{2\pi})^t(\eta + \frac{\eta_\theta}{2\pi})\big). \tag{4.12}$$

We write $\tilde{\mathcal{Q}}(z) := \tilde{\mathcal{Q}}\circ\mathcal{I}^{-1}(z)$.

Definition 4.7. *The Bargmann transform of a function $f \in L^2(\mathcal{E},\mathbb{C})$ is a function $\mathcal{B}\,f : \mathcal{E}\oplus\mathcal{E}^\vee \to \mathbb{C}$ defined fiberwise by*

$$(\mathcal{B}\,f)|_{(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}}(W,\eta) := \int_{\mathcal{E}_{(x,y,\theta)}} f|_{\mathcal{E}_{(x,y,\theta)}}(V)\, e^{2\pi V\bullet z - \pi V^t A_{(x,y)} V + \frac{\pi}{2}\mathcal{Q}(z)}\,\mathrm{dvol}_{(x,y,\theta)}(V)\,, \tag{4.13}$$

with the notation as in (4.11) and with $\mathrm{dvol}_{(x,y,\theta)}(V)$ the volume form on the fibers of $\mathcal{E}$ determined by the Riemannian metric on S.

Lemma 4.8. *Consider the window function Ψ_0 as in (3.6). The Gabor functions*

$$\rho(W,\eta)\Psi_0(V) = e^{2\pi i\langle \eta, V-W\rangle}\Psi_0(V-W)\,,$$

with $(W,\eta)\in\mathcal{E}\bigoplus\mathcal{E}^\vee$, satisfy

$$|\langle f, \rho(W,\eta)\Psi_0\rangle| = |\mathcal{B}\,f(W - i(\eta + \frac{\eta_\theta}{2\pi}))|\, e^{-\tilde{\mathcal{Q}}(W,\eta)}\,. \tag{4.14}$$

with $\tilde{\mathcal{Q}}$ as in (4.12).

Proof. We have

$$\begin{aligned}
\langle f, \rho(W,\eta)\Psi_0\rangle &= \int_{\mathcal{E}_{(x,y,\theta)}} f(V)\, e^{-\pi(V-W)^t \frac{A}{\pi}(V-W) - 2\pi i\langle \frac{\eta_\theta}{2\pi}, V-W\rangle} e^{-2\pi i\langle \eta, V\rangle}\,\mathrm{dvol}(V)\\
&= e^{2\pi i\langle \frac{\eta_\theta}{2\pi}, W\rangle}\int_{\mathcal{E}_{(x,y,\theta)}} f(V)\, e^{\pi V^t\frac{A}{\pi}V - 2\pi V^t\frac{A}{\pi}W - \pi W^t\frac{A}{\pi}W} e^{-2\pi i\langle\frac{\eta_\theta}{2\pi}+\eta, V\rangle}\,\mathrm{dvol}(V)\\
&= e^{2\pi i\langle \frac{\eta_\theta}{2\pi}, W\rangle} e^{-i\pi\langle \eta + \frac{\eta_\theta}{2\pi}, W\rangle} e^{-\frac{\pi}{2}W^t\frac{A}{\pi}W + \frac{\pi}{2}(\eta+\frac{\eta_\theta}{2\pi})^t\cdot(\eta+\frac{\eta_\theta}{2\pi})}\cdot\\
&\cdot\int_{\mathcal{E}_{(x,y,\theta)}} f(V)\, e^{-2\pi V\bullet(W - i(\frac{\eta_\theta}{2\pi}+\eta))} e^{-\pi V^t\frac{A}{\pi}V} e^{-\frac{\pi}{2}\mathcal{Q}(W - i(\eta+\frac{\eta_\theta}{2\pi}))}\mathrm{dvol}(V)
\end{aligned}$$

with $\mathcal{Q}$ as in (4.10) and $\tilde{\mathcal{Q}}$ as in (4.12). ∎

Remark 4.9. Under the identification (4.9) we write (4.14) equivalently as

$$|\langle f, \rho(W,\eta)\Psi_0\rangle| = |\mathcal{B}f(\overline{z})| e^{-\tilde{\mathcal{Q}}(W,\eta)} \quad \text{for } z = W + i(\frac{\eta_\theta}{2\pi} + \eta)\,. \tag{4.15}$$

Definition 4.10. *The global Bargmann-Fock space $\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)$ is the space of functions $F : \mathcal{E}\oplus\mathcal{E}^\vee \to \mathbb{C}$ such that $F|_{(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}}\circ\mathcal{I}^{-1} : \mathbb{C}^2 \to \mathbb{C}$ is entire with*

$$\|F\|^2_{\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)} = \int_M\int_{\mathbb{C}^2}\Big|F|_{(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}}\circ\mathcal{I}^{-1}(z)\Big|^2 e^{-2\tilde{\mathcal{Q}}(z)}dz\,\mathrm{dvol}(x,y,\theta) < \infty\,.$$

The fiberwise Bargmann-Fock space $\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}$ is the space of functions $F : (\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)} \to \mathbb{C}$ such that $F\circ\mathcal{I}^{-1} : \mathbb{C}^2 \to \mathbb{C}$ is entire, with the norm

$$\|F\|^2_{\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}} = \int_{\mathbb{C}^2}\Big|F|_{(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}}\circ\mathcal{I}^{-1}(z)\Big|^2 e^{-2\tilde{\mathcal{Q}}(z)}dz < \infty\,.$$

The space $\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)$ is a Hilbert space with the inner product

$$\langle F,G\rangle_{\mathcal{F}^2} := \int_M\int_{\mathbb{C}^2} F|_{(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}}\circ\mathcal{I}^{-1}(z)\,\overline{G|_{(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}}\circ\mathcal{I}^{-1}(z)}\,e^{-2\tilde{\mathcal{Q}}(z)}\,dz\,d\mathrm{vol}(x,y,\theta)\,.$$

Indeed, $\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)$ is the direct integral over $(M,d\mathrm{vol})$ of a family of Hilbert spaces $\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}$, which are isomorphic, through the map $\mathcal{I}$ of (4.9) with the Hilbert space $L^2(\mathbb{C}^2,e^{-2\tilde{\mathcal{Q}}(z)}dz)$.

In this geometric setting we formulate the sampling condition in the following way.

Definition 4.11. *Let Λ be a bundle of lattices over M where, over a local chart we have $\Lambda_{(x,y,\theta)}$ a lattice in $(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}$. The bundle Λ satisfies the smooth sampling condition for $\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)$ if there are $\mathbb{R}^*_+$-valued smooth functions C,C' on the local charts of M, such that, for all (x,y,θ) in a local chart of M and for all $F\in\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}$, the estimates*

$$C_\mu\cdot\|F\|^2_{\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)_\mu}\le\sum_{(W,\eta)\in\Lambda_\mu}\left|F|_{(\mathcal{E}\oplus\mathcal{E}^\vee)_\mu}\right|^2 e^{-2\tilde{\mathcal{Q}}_\mu(W,\eta)}\le C'_\mu\cdot\|F\|^2_{\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)_\mu}\tag{4.16}$$

are satisfied, for $\mu=(x,y,\theta)$ in a local chart of M, and with $\tilde{\mathcal{Q}}$ as in (4.12).

Lemma 4.12. *For any (x,y,θ) in a local chart of M, the Bargmann transform $\mathcal{B}$ of (4.13) is a bijection from $L^2(\mathcal{E}_{(x,y,\theta)})$ to $\mathcal{F}^2(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}$, with*

$$\|\mathcal{B}f\|_{\mathcal{F}(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}}=K_{(x,y)}\cdot\|f\|_{L^2(\mathcal{E}_{(x,y,\theta)})}\,,\tag{4.17}$$

for a smooth $\mathbb{R}^*_+$-valued function K over the local charts U of S. Moreover, $\mathcal{G}(\Psi_0,\Lambda)$ is a frame for $L^2(\mathcal{E}_{(x,y,\theta)})$ if and only if $\overline{\Lambda}+i\frac{\eta_\theta}{2\pi}$ is a set of sampling for $\mathcal{F}(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}$.

Proof. For the window function Ψ_0 as in (3.6), we have

$$\|\Psi_0\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})}=\int_{\mathcal{E}_{(x,y,\theta)}}|\Psi_0(V)|^2\,dV=\int_{\mathcal{E}_{(x,y,\theta)}}e^{-2V^tA_{(x,y)}V}\,dV=\frac{\pi}{2\sqrt{\det(A_{(x,y)})}}\,.$$

as a standard Gaussian integral in 2-dimensions. Because we assumed that the matrices $A_{(x,y)}$ in the window function Ψ_0 of (3.6) have spectrum bounded away from zero, and that S is compact, the quantity

$$K_{(x,y)}:=\frac{\pi}{2\sqrt{\det(A_{(x,y)})}}$$

determines a smooth real valued function $K:S\to\mathbb{R}$ with a strictly positive minimum and a bounded maximum. Moreover, by Theorem 3.2.1 and Corollary 3.2.2 of [22], the orthogonality relation

$$\langle V_{\phi_1}f_1,V_{\phi_2}f_2\rangle_{L^2(\mathbb{R}^{2n})}=\langle f_1,f_2\rangle_{L^2(\mathbb{R}^n)}\cdot\overline{\langle\phi_1,\phi_2\rangle_{L^2(\mathbb{R}^n)}}$$

for the short time Fourier transform

$$V_\phi f(x,\omega)=\int_{\mathbb{R}^n}f(t)\,\overline{\phi(t-x)}\,e^{-2\pi it\cdot\omega}\,dt\,,\quad\text{with }\ (x,\omega)\in\mathbb{R}^{2n}\,,$$

gives the identity

$$\|\langle f,\rho(W,\eta)\Psi_0\rangle\|_{L^2(\mathcal{E}_{(x,y,\theta)})}=\|f\|_{L^2(\mathcal{E}_{(x,y,\theta)})}\cdot\|\Psi_0\|_{L^2(\mathcal{E}_{(x,y,\theta)})}\,.$$

Moreover, by (4.15) we have, for $z=\mathcal{I}(W,\eta)$,

$$\|\langle f, \rho(W,\eta)\Psi_0\rangle\|_{L^2(\mathcal{E}_{(x,y,\theta)})} = \int_{(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}} |\langle f, \rho(W,\eta)\Psi_0\rangle|^2 d\mathrm{vol}(W,\eta)$$
$$= \int_{\mathbb{C}^2} |\mathcal{B}\, f(\bar{z})|^2 e^{-2\tilde{Q}(z)}\, dz\,.$$

Injectivity then follows, while surjectivity follows by the same argument showing the density of $\mathcal{B}(L^2(\mathbb{R}^n)) \subset \mathcal{F}^2_n$ in the proof of Theorem 3.4.3 of [22], applied pointwise in $(x,y,\theta) \in M$.

The Gabor system $\mathcal{G}(\Psi_0, \Lambda)$ satisfies the smooth frame condition of Definition 4.1 if there are smooth functions $C_{(x,y,\theta)}, C'_{(x,y,\theta)} > 0$ on the local charts of M such that

$$C_{(x,y,\theta)}\, \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})} \leq \sum_{\lambda=(W,\eta)\in\Lambda} |\langle f, \rho(\lambda)\Psi_0\rangle|^2 \leq C'_{(x,y,\theta)} \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})}\,.$$

By (4.15) we see that this is equivalent to the smooth sampling condition of Definition 4.11 for $\overline{\Lambda} + i\frac{\eta_\theta}{2\pi}$. ☐

Proposition 4.13. *With the scaling by the function $b = b_M(x,y,\theta)$ of (3.12), the Gabor system $\mathcal{G}(\Psi_0, \Lambda_{b,\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ satisfies the frame condition.*

Proof. We write here the window function Ψ_0 as Ψ_0^A to emphasize the dependence on the quadratic from $A = A_{(x,y)}$. Let $f : \mathcal{E} \to \mathbb{R}$ be a signal, with $f|_{\mathcal{E}_{(x,y,\theta)}} \in L^2(\mathcal{E}_{(x,y,\theta)})$. We have

$$\sum_{\lambda\in\Lambda_{b,\alpha,J}\oplus\Lambda^\vee_{\alpha,J}} \left|\langle f, \rho(\lambda)\Psi_0^A\rangle\right|^2$$
$$= \sum_{(n,m)\in\mathbb{Z}^2\times\mathbb{Z}^2} \left| \int_{\mathcal{E}_{(x,y,\theta)}} f(V) e^{2\pi i m\cdot V} e^{(V-bn)^t A_{(x,y,\theta)}(V-bn) + i\langle \eta_\theta, V-bn\rangle} d\mathrm{vol}_{(x,y,\theta)}(V)\right|^2 .$$

With a change of variables $V = \sqrt{b}\,U$ and correspondingly changing the quadratic form A to bA, we rewrite the above as

$$\sum_{(n,m)\in\mathbb{Z}^2\times\mathbb{Z}^2} b^2 \left| \int_{\mathcal{E}_{(x,y,\theta)}} f(\sqrt{b}U)\overline{M_{\sqrt{b}m} T_{\sqrt{b}n} e^{-U(bA_{(x,y,\theta)})U^T - i\langle \eta_\theta\sqrt{b}, U\rangle}} d\mathrm{vol}_{(x,y,\theta)}(V)\right|^2$$
$$= b^2 \sum_{\tilde{\lambda}\in\sqrt{b}\Lambda_{\alpha,J}\oplus\sqrt{b}\Lambda^\vee_{\alpha,J}} \left|\langle f_{\sqrt{b}}, \rho(\tilde{\lambda})\Psi_0^{bA}\rangle\right|^2,$$

where $f_{\sqrt{b}}(V) = f(\sqrt{b}V)$. Therefore, the Gabor system $\mathcal{G}(\Psi_0^A, \Lambda_{b,\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ is a frame for $L^2(\mathcal{E}_{(x,y,\theta)})$ if and only if $\mathcal{G}(\Psi_0^{bA}, \Lambda_{\sqrt{b},\alpha,J} \oplus \Lambda^\vee_{\sqrt{b},\alpha,J})$ is a frame for $L^2(\mathcal{E}_{(x,y,\theta)})$. Moreover, by Lemma 4.12, we know that $\mathcal{G}(\Psi_0^{bA}, \Lambda_{\sqrt{b},\alpha,J} \oplus \Lambda^\vee_{\sqrt{b},\alpha,J})$ is a frame for $L^2(\mathcal{E}_{(x,y,\theta)})$ if and only if the uniformly discrete set $(\sqrt{b}\mathbb{Z}^2 + i\sqrt{b}\mathbb{Z}^2) + i\frac{\eta_\theta}{2\pi}$ is a set of sampling for $\mathcal{F}(\mathcal{E}\oplus\mathcal{E}^\vee)_{(x,y,\theta)}$. Finally, by [24], $(\sqrt{b}\mathbb{Z}^2 + i\sqrt{b}\mathbb{Z}^2) + i\frac{\eta_\theta}{2\pi}$ is a set of sampling if and only if the complex lattice $T(\mathbb{Z}^2 + i\mathbb{Z}^2)$ is a set of sampling, for the matrix

$$T = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & \sqrt{b} \end{pmatrix}.$$

By Proposition 11 of [24], the latter condition is satisfied if and only if $\sqrt{b} < 1$, which we know is the case by Lemma 3.4. ☐

5 Gabor frames: symplectization and contactization

As in Section 2.3, we consider the contactization $\mathcal{CS}(M)$ of the symplectization $\mathcal{S}(M)$ of the manifold of contact elements $M = \mathbb{S}_w(T^*S)$ of a surface S. This model is motivated by the goal of describing visual perception based on neurons sensitive not only to orientation, but also to frequency and phase, with the frequency-phase and the position-orientation uncertainty minimized by the Gabor functions profiles. From the point of view of this model, it is worth pointing out that, although higher dimensional, the 5-dimensional contact manifold $\mathcal{CS}(M)$ is completely determined by the contact 3-manifold M with no additional independent choices, being just the contactization of the symplectization.

Note that, while the contact structure of $\mathcal{CS}(M)$ is the natural extension of the contact structure of M, this does not directly imply that modelling the visual cortex requires an increasing family of contact structures to account for different families of cells sensitive to different features, as different features may be described by the same geometry.

Given local charts on M with the choice of local basis

$$X_\theta = \partial_\theta, \quad R_{\alpha_J} = -w^{-1}\sin(\theta)\partial_x + w^{-1}\cos(\theta)\partial_y \tag{5.1}$$

for the contact planes ξ of the contact structure α on M and the Reeb field $R_\alpha = w^{-1}\cos(\theta)\partial_x + w^{-1}\sin(\theta)\partial_y$, we obtain a basis of the contact hyperplane distribution of the five-dimensional contact manifold $(T^*S_0 \times S^1, \tilde{\alpha})$, in the corresponding local charts, given by

$$\{X_\theta, R_{\alpha_J}, R_{\phi,\alpha}, X_w\},$$
$$\text{with} \quad R_{\phi,\alpha} := \partial_\phi + R_\alpha \quad \text{and} \quad X_w := \partial_w\,.$$

In the case of the twisted contact structure α_J, with the choice of basis

$$X_\theta = \partial_\theta, \quad R_\alpha = w^{-1}\cos(\theta)\partial_x + w^{-1}\sin(\theta)\partial_y \tag{5.2}$$

for the contact plane distribution ξ_J, and the Reeb vector field $R_{\alpha,J} = -w^{-1}\sin(\theta)\partial_x + w^{-1}\cos(\theta)\partial_y$, we similarly obtain a basis for the contact hyperplanes $\tilde{\xi}_J$ given by

$$\begin{aligned} &\{X_\theta, R_\alpha, R_{\phi,\alpha,J}, X_w\}, \\ &R_{\phi,\alpha,J} := \partial_\phi + R_{\alpha,J}\,. \end{aligned} \tag{5.3}$$

The bundle $\mathcal{E}$ of signal planes on M determines the following bundles on the symplectization $\mathcal{S}(M)$ and the contactization $\mathcal{CS}(M)$.

Definition 5.1. *Let $\hat{\mathcal{E}}$ denote the pull-back of the bundle $\mathcal{E}$ of signal planes to $T^*S_0 \simeq M \times \mathbb{R}_+^*$ via the projection to M, and let $\tilde{\mathcal{E}}$ denote the vector bundle over $\mathcal{CS}(M)$ given by $\hat{\mathcal{E}} \boxplus TS^1 = \pi^*_{T^*S_0}\hat{\mathcal{E}} \oplus \pi^*_{S^1}TS^1$, with pull-backs taken with respect to the two projections of $\mathcal{CS}(M) = T^*S_0 \times S^1$ on the two factors.*

The signals in this setting will be functions $I : \tilde{\mathcal{E}} \to \mathbb{R}$. The vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{CS}(M)$ is a rank 3 real vector bundle over a 5-dimensional manifold.

Remark 5.2. A basis of sections for $\tilde{\mathcal{E}}$ over a local chart is obtained by taking the vectors $\{R_\alpha, R_{\alpha_J}, \partial_\phi\}$. There are two other choices of basis directly determined by the contact forms $\tilde{\alpha}$ and $\tilde{\alpha}_J$, namely $\{R_\alpha, R_{\phi,\alpha,J}, R_{\tilde{\alpha}_J}\}$, where the first two vectors span the intersection $\tilde{\mathcal{E}} \cap \tilde{\xi}$ of the contact hyperplane distribution with the bundle $\tilde{\mathcal{E}}$ and the last vector is the Reeb field of $\tilde{\alpha}_J$, or $\{R_{\alpha_J}, R_{\phi,\alpha}, R_{\tilde{\alpha}}\}$, with the first two vector fields spanning $\tilde{\mathcal{E}} \cap \tilde{\xi}_J$ and the third the Reeb field of $\tilde{\alpha}$. The first basis has the advantage of a providing consistent choices of basis for both $\mathcal{E}$ and $\tilde{\mathcal{E}}$.

Lemma 5.3. *In a local chart of $\mathcal{S}(M)$ with coordinates (x, y, w, θ), the window function Ψ_0 as in (3.6) extends to a window function on $\hat{\mathcal{E}}$ given by*

$$\hat{\Psi}_{0,(x,y,w,\theta)}(V) = \exp\left(-V^t A_{(x,y)} V - i\langle \eta_{(w,\theta)}, V\rangle_{(x,y)}\right), \tag{5.4}$$

with $\eta_{(w,\theta)} = (w\cos(\theta), w\sin(\theta))$.

Proof. The window function Ψ_0 as in (3.6) is obtained as restriction to $TS \oplus \mathbb{S}(T^*S)$ of a window function Φ_0 on $TS \oplus T^*S$ defined as in (3.5) in Definition 3.1. By identifying $\mathcal{S}(M) = T^*S_0$ and $\hat{\mathcal{E}}$ with the pull-back of TS to $\mathcal{S}(M)$, we see that Φ_0 induces a window function $\hat{\Psi}_0$ on $\hat{\mathcal{E}}$ of the form (5.4). □

We further extend the window function (5.4) to $\tilde{\mathcal{E}}$ so as to obtain a window function that is a modified form of the function considered in the model of [5].

Definition 5.4. *In a local chart of $\mathcal{CS}(M)$ with coordinates (x, y, w, θ, ϕ), window functions on $\tilde{\mathcal{E}}$ extending the window function (5.4) are functions on $\tilde{\mathcal{E}}$ of the form*

$$\tilde{\Psi}_{0,(x,y,w,\theta,\phi),\zeta_0}(V, \upsilon) = \exp\left(-V^t A_{(x,y)} V - i\langle \eta_{(w,\theta)}, V\rangle_{(x,y,w,\theta)} - \kappa_\phi^2 \upsilon^2 - i\langle \zeta_0, \upsilon\rangle_\phi\right), \tag{5.5}$$

for $\eta_{(w,\theta)}$ as in (5.4), and with $\zeta_0 \in T^*_\phi S^1$ and $\upsilon \in T_\phi S^1$. The two-dimensional Gabor systems of the form $\{\rho(W, \eta)\Psi_0|_{\mathcal{E}_{(x,y,\theta)}}\}$ are then replaced by a three-dimensional system of the form

$$\rho(W, \eta, \nu, \zeta)\tilde{\Psi}_0|_{\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}}(V, \upsilon), \tag{5.6}$$

with $(W, \eta, \nu, \zeta) \in (\tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee)_{(x,y,w,\theta,\phi)}$.

In the setting of [5], the additional variables $\phi \in S^1$ (with its linearization $\upsilon \in T_\phi S^1$) and the dual variable $\zeta \in T^*_\phi S^1$, which we view here as part of the bundle $\tilde{\mathcal{E}}$ over the contact manifold $\mathcal{CS}(M)$, represent a model of phase and velocity of spatial wave propagation. The window function $\tilde{\Psi}_0$ that we consider here differs from the function considered in [5], which does not have the Gaussian term in the $\upsilon \in T_\phi S^1$ variable. While they consider the limit case where $\kappa_\phi = 0$, we argue here that one needs this additional term to be non-zero (though possibly small) in order to have good signal analysis properties for the associated Gabor system, in the presence of these additional variables. The Gaussian term in υ can in principle be replaced by another rapid decay function, however, it seems more natural to use a Gaussian term, like we have for the variables in $\mathcal{E}$, in order to maintain a similar structure for all the variables of $\tilde{\mathcal{E}}$. We will return to discuss the case $\kappa_\phi = 0$ of [5] in Section 5.1.

Note that the goal of the model of [5] is different, as they apply the Gabor transform to signals that are independent of the frequency and phase variables, so that the problem outlined above with the frame condition does not arise. It is only when the signal analysis is performed on the larger space given by the 3-dimensional linear fibers of the bundle $\tilde{\mathcal{E}}$, rather than on the 2-dimensional bundle $\hat{\mathcal{E}}$, that one needs to modify the window function as described above.

Let $\tilde{\mathcal{E}}^\vee$ denote the dual bundle of $\tilde{\mathcal{E}}$, with the choice of local basis $\{R_\alpha, R_{\alpha,J}, \partial_\phi\}$ for $\tilde{\mathcal{E}}$ and the dual local basis $\{\alpha, \alpha_J, d\phi\}$. This determines bundles of framed lattices over the local charts of $\mathcal{CS}(M)$

$$\tilde{\Lambda} = \mathbb{Z} R_\alpha + \mathbb{Z} R_{\alpha_J} + \mathbb{Z}\partial_\phi = \Lambda_{\alpha,J} \oplus L, \quad \tilde{\Lambda}^\vee = \mathbb{Z}\alpha + \mathbb{Z}\alpha_J + \mathbb{Z}\, d\phi = \Lambda^\vee_{\alpha,J} \oplus L^\vee, \tag{5.7}$$

with $\Lambda_{\alpha,J}$ and $\Lambda^\vee_{\alpha,J}$ the bundles of framed lattices of (3.8) and (3.9).

We consider the bundle of framed lattices $\tilde{\Lambda} \oplus \tilde{\Lambda}^\vee$, which has the property that, in a local chart, the fibers

$$(\tilde{\Lambda} \oplus \tilde{\Lambda}^\vee)_{(x,y,w,\theta,\phi)} \subset (\tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee)_{(x,y,w,\theta,\phi)}$$

are lattices in the fibers of the 3-plane bundle $\tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee$ over the 5-dimensional contact manifold $\mathcal{CS}(M)$.

The window function $\tilde{\Psi}_0$ and the bundle of framed lattices $\tilde{\Lambda} \oplus \tilde{\Lambda}^\vee$ determine a Gabor system

$$\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee) = \left\{ \rho(\lambda)\tilde{\Psi}_0|_{\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}} \,\middle|\, \lambda = (W, \eta, \nu, \zeta) \in (\tilde{\Lambda} \oplus \tilde{\Lambda}^\vee)_{(x,y,w,\theta,\phi)} \right\} . \tag{5.8}$$

As in the case of the bundle of framed lattices $\Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J}$ we consider a scaling of the lattices in the fibers of $\tilde{\Lambda} \oplus \tilde{\Lambda}^\vee$, for the same reasons discussed in Section 3.5. We define $R_{\max} > 0$ as in Section 3.5. For the TS^1 direction of $\tilde{\mathcal{E}}$, the injectivity radius $R^{S^1}_{inj}$ is constant and equal to half the length of the S^1 circle. Thus, we take, as in Section 3.5, a scaling factor of the form $\gamma = R^{S^1}_{inj}/R_{\max}$. As discussed in Section 3.5 we can assume that in our model $R_{\max} > R^{S^1}_{inj}$ so that $\gamma < 1$. We then consider the bundle of framed lattices determined by this choice of scaling on L and the previous choice of scaling on $\Lambda_{\alpha,J}$.

Definition 5.5. *Let $\tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee$ be the bundle of framed lattices of the form*

$$\tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee = \Lambda_{b,\alpha,J} \oplus L_\lambda \oplus \Lambda^\vee_{\alpha,J} \oplus L^\vee , \tag{5.9}$$

where $\Lambda_{b,\alpha,J}$ is the scaled lattice of (3.13) with $b = b_M$ the function of (3.12), while $L_\lambda = \lambda L$ for the constant $\lambda = R^{S^1}_{inj}/R_{\max}$ as above. This has associated Gabor system

$$\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee) = \left\{ \rho(\lambda)\tilde{\Psi}_0 \,\middle|\, \lambda = (W, \eta, \nu, \zeta) \in \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee \right\} , \tag{5.10}$$

where for simplicity of notation we have suppressed the explicit indication of the fibers of $\tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee$ as in (5.8).

We then have the following result about the Gabor frame condition for the Gabor systems (5.8) and (5.10).

Proposition 5.6. *The Gabor system $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee)$ of (5.8) is not a frame. The Gabor system $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee)$ of (5.10) is a frame.*

Proof. By construction the Gabor systems with window function $\tilde{\Psi}_0$ and lattice $\tilde{\Lambda} \oplus \tilde{\Lambda}^\vee$ or $\tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee$ split as a product of a 2-dimensional system $\mathcal{G}(\Psi_0, \Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ or $\mathcal{G}(\Psi_0, \Lambda_{b,\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ and a 1-dimensional Gabor system $\mathcal{G}(\psi_0, L \oplus L^\vee)$ or $\mathcal{G}(\psi_0, L_\lambda \oplus L^\vee)$, where

$$\psi_{0,\phi}(\upsilon) = \exp(-\kappa_\phi \upsilon^2 - i\langle \zeta_0, \upsilon \rangle_\phi) = \exp(-\kappa_\phi \upsilon^2 - i\zeta_0 \upsilon) .$$

Thus, the frame condition for $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee)$ holds if and only if it holds for both $\mathcal{G}(\Psi_0, \Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ and $\mathcal{G}(\psi_0, L \oplus L^\vee)$ and similarly the frame condition for $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee)$ holds if and only if it holds for both $\mathcal{G}(\Psi_0, \Lambda_{b,\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ and $\mathcal{G}(\psi_0, L_\lambda \oplus L^\vee)$. For the 1-dimensional systems with a rapid decay function as window function, the frame condition holds if and only if the lower Beurling density D^- of the lattice is strictly greater than one. For the lattice $L \oplus L^\vee$ this condition is not satisfies (see Proposition 4.3) so the Gabor system is not a frame, while for the lattice $L_\lambda \oplus L^\vee$ is satisfied since $\gamma < 1$ (see Proposition 4.4). Thus, in the case of the Gabor system $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee)$ of (5.10) the question is reduced to the question of whether the 2-dimensional system $\mathcal{G}(\Psi_0, \Lambda_{b,\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ is a frame. We know this system is indeed a frame by Proposition 4.13. ☐

5.1 Gelfand triples and Gabor frames

We return here to discuss the case of the profiles considered in [5], with the term $\kappa_\phi = 0$. As mentioned above, the function $\tilde{\Psi}_0|_{\kappa_\phi=0}$ is not a window function for a Gabor system in the usual sense, as it is not of rapid decay (and not even L^2) along the fibers of $\tilde{\mathcal{E}}$. However, we can still interpret it as a tempered distribution on the fibers of $\tilde{\mathcal{E}}$. Thus, one can at least ask the question of whether this window function defines Gabor frames in a distributional sense. To formulate Gabor systems in such a setting, it is convenient to consider the formalism of Gelfand triples (also known as rigged Hilbert spaces, [21]).

We consider here the same setting as in [40, 41] for distributional frames, with Gelfand triples given by

$$\mathcal{S}(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}) \hookrightarrow L^2(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}, \mathrm{dvol}_{(x,y,w,\theta,\phi)}) \hookrightarrow \mathcal{S}'(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}) \,,$$

where the space $\mathcal{S}$ of tempered distributions is densely and continuously embedded in the L^2-Hilbert space, which is densely and continuously embedded in the dual space $\mathcal{S}'$ of distributions. The pairing $\langle f, g\rangle$ of distributions $f \in \mathcal{S}'$ and test functions $g \in \mathcal{S}$ extends the Hilbert space inner product. We write the above triples for simplicity of notation in the form

$$\mathcal{S}(\tilde{\mathcal{E}}) \hookrightarrow L^2(\tilde{\mathcal{E}}) \hookrightarrow \mathcal{S}'(\tilde{\mathcal{E}}) \,.$$

Definition 5.7. *A distributional Gabor system $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda})$ on $\tilde{\mathcal{E}}$ is given by a window generalized-function $\tilde{\Phi}_0 \in \mathcal{S}'(\tilde{\mathcal{E}})$ and a bundle of lattices $\tilde{\Lambda}$ with*

$$\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda}) = \{\rho(\lambda)\tilde{\Phi}_0 \,|\, \lambda \in \tilde{\Lambda}\} \subset \mathcal{S}'(\tilde{\mathcal{E}}) \,.$$

The distributional Gabor system $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda})$ is a distributional frame for the bundle $\tilde{\mathcal{E}}$ on $\mathcal{CS}(M)$ if there are bounded smooth functions $C, C' : \mathcal{CS}(M) \to \mathbb{R}_+^*$ with strictly positive $\inf_{\mathcal{CS}(M)} C$ and $\inf_{\mathcal{CS}(M)} C'$, such that, for all $f \in \mathcal{S}(\tilde{\mathcal{E}})$

$$C_{(x,y,w,\theta,\phi)} \, \|f\|^2_{L^2(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)})} \leq \sum_{\lambda \in \tilde{\Lambda}_{(x,y,w,\theta,\phi)}} |\langle \rho(\lambda)\tilde{\Phi}_0, f\rangle|^2 \leq C'_{(x,y,w,\theta,\phi)} \, \|f\|^2_{L^2(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)})} \,.$$

Lemma 5.8. *Let $\tilde{\Phi}_0 = \tilde{\Psi}_0|_{\kappa_\phi=0}$, with $\tilde{\Psi}_0$ as in (5.5). The systems $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee)$ and $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee)$ with the lattices as in Definition 5.5, are distributional Gabor systems that decompose into a product of a 2-dimensional ordinary Gabor system given by $\mathcal{G}(\Psi_0, \Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$ or $\mathcal{G}(\Psi_0, \Lambda_{b,\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$, respectively, and a 1-dimensional distributional Gabor system of the form $\mathcal{G}(\phi_0, L \oplus L^\vee)$ or $\mathcal{G}(\phi_0, L_\gamma \oplus L^\vee)$, respectively, with window generalized-function $\phi_0(\upsilon) = \exp(-i\zeta_0\upsilon) \in \mathcal{S}'(\mathbb{R})$. The distributional Gabor system $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee)$ does not satisfy the distributional Gabor frame condition. The distributional Gabor system $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee)$ satisfies the distributional Gabor frame condition if and only if the 1-dimensional distributional Gabor system $\mathcal{G}(\phi_0, L_\gamma \oplus L^\vee)$ satisfies the distributional frame condition.*

Proof. We view $\tilde{\Phi}_0$ as the distribution in $\mathcal{S}'(\tilde{\mathcal{E}})$ that acts on test functions $f \in \mathcal{S}(\tilde{\mathcal{E}})$ as

$$\langle f, \tilde{\Phi}_0\rangle_{(x,y,w,\theta,\phi)} = \int_{\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}} \overline{\tilde{\Phi}_0|_{\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}}(V,\upsilon)} \, f(V,\upsilon) \, \mathrm{dvol}_{(x,y,w,\theta,\phi)}(V,\upsilon) \,.$$

As in Proposition 5.6 we see that the distributions $\rho(\lambda)\tilde{\Phi}_0$ are products of a function $\rho(\lambda')\Psi_0 \in \mathcal{S}(\tilde{\mathcal{E}})$ and a distribution $\rho(\lambda'')\phi_0$ in $\mathcal{S}'(\tilde{\mathcal{E}})$, with $\lambda = (\lambda', \lambda'')$ for $\lambda \in \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee$ and $\lambda' \in \Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J}$ and $\lambda'' \in L \oplus L^\vee$ (and similarly for the scaled versions of the lattices). Since these Gabor systems decouple, the distributional frame condition becomes equivalent to the ordinary frame condition for the part that is an ordinary frame and the distributional frame condition for the part that is a distributional frame. Thus, the distributional Gabor systems $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee)$ and $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee)$ are distributional Gabor frames if and only if the respective 2-dimensional ordinary Gabor systems are ordinary frames and the respective 1-dimensional distributional Gabor systems are distributional frames. In the first case we know that the frame condition already fails at the level of the 2-dimensional ordinary Gabor system. In the second case the 2-dimensional system satisfies the usual frame condition by Proposition 4.13, hence the question reduces to whether the 1-dimensional distributional system $\mathcal{G}(\phi_0, L_\gamma \oplus L^\vee)$ satisfies the distributional frame condition. ☐

The following statement shows that, even when interpreted in this distributional setting the Gabor system generated by the window function $\tilde{\Phi}_0$ as in [5] does not give rise to frames, hence it does not allow for good signal analysis.

Proposition 5.9. *The distributional Gabor system $\mathcal{G}(\tilde{\Phi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee)$ does not satisfy the distributional frame condition.*

Proof. By Lemma 5.8 we can equivalently focus on the question of whether the one-dimensional distributional Gabor system $\mathcal{G}(\phi_0, \gamma\mathbb{Z} + \mathbb{Z})$ satisfies the distributional frame condition. Given a signal $f \in \mathcal{S}(\mathbb{R})$, we have, for $\lambda = (\gamma n, m)$ and $\phi_0(t) = e^{-i\zeta_0 t}$,

$$\begin{aligned}\langle f, \rho(\lambda)\phi_0\rangle &= \int_{\mathbb{R}} e^{-2\pi i m t} f(t) e^{i\zeta_0(t-\gamma n)}\, dt \\ &= e^{-i\zeta_0\gamma n} \int_{\mathbb{R}} e^{-2\pi i (m - \frac{\zeta_0}{2\pi})t} f(t)\, dt = e^{-i\zeta_0\gamma n} \hat{f}(m - \frac{\zeta_0}{2\pi})\,.\end{aligned}$$

Note that when we take $|\langle f, \rho(\lambda)\phi_0\rangle|^2$ the dependence on n disappears entirely so the sum over the lattice is always divergent. ☐

Remark 5.10. The window function $\tilde{\Psi}_0|_{\kappa_\phi=0}$ in [5] is chosen so that the Lie group and Lie algebra structure underlying receptive profiles of this form (see [33, 37]) determines horizontal vector fields given by the basis $\{X_\theta, R_\alpha, R_{\phi,\alpha,J}, X_w\}$ of (5.3) of the contact hyperplanes $\tilde{\xi}_J$. However, if we replace this choice of window with our window $\tilde{\Psi}_0$ where $\kappa_\phi \neq 0$, the same Lie group of transformations acts on these types of profiles generating the same horizontal vector fields. Note that also the original goal of [5] of describing receptive profiles of neurons sensitive to frequency and phase variables, with the frequency-phase uncertainty minimized is already satisfied by the Gabor system generated by our proposed window function $\tilde{\Psi}_0$, without the need to impose $\kappa_\phi = 0$.

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